Name:____________________________________ Advanced 1999

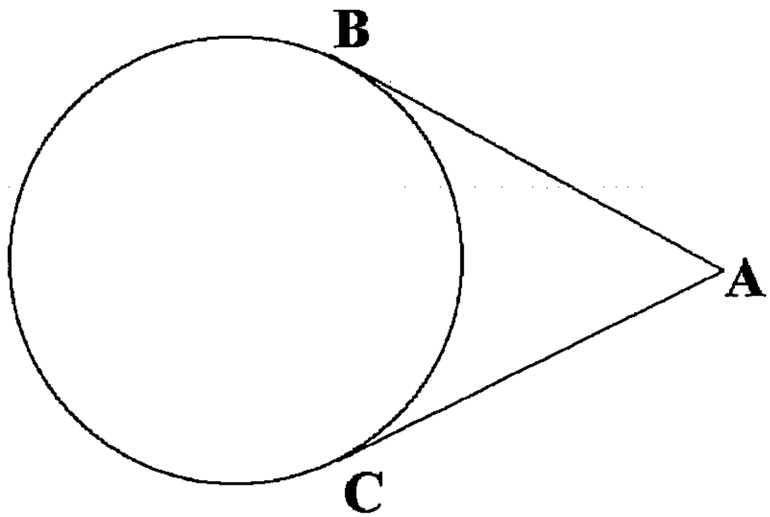
School:__________ # 1

Let a, b, and c be fixed numbers, with neither a nor c equal to zero. Define functions f(x) and g(x) as follows: $f(x) = ax^2 + bx + c$ and $g(x) = cx^2 + bx + a$. Prove the following: if $x = r$ is a root of f(x), then $x = 1/r$ is a root of g(x).

Name:__________________________________ Advanced 1999

School:__________ # 2

In the figure at the right, line segments AB and AC are tangent to the circle, with points of tangency A and B respectively. If the radius of the circle is five inches and the measure of angle BAC is $60°$, find the distance between points B and C. Find the exact distance and the approximate distance accurate to four decimal places.



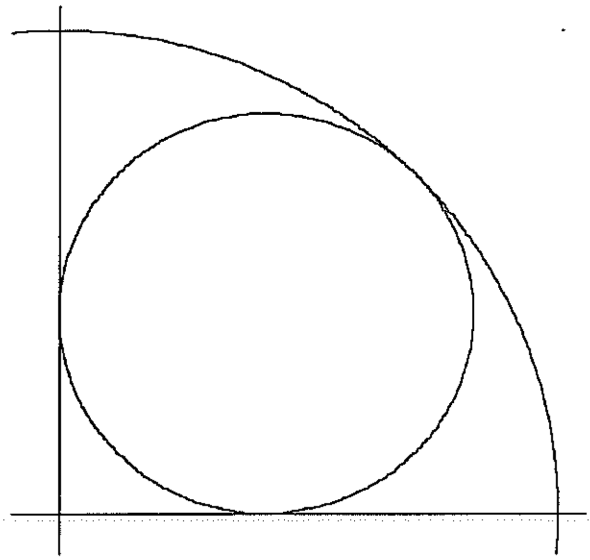
Name:______________________________ Advanced 1999
School:__________ # 3

Find a cubic polynomial (a polynomial of degree three) having real coefficients so that the numbers $x = 2$ and $x = 3 + \boldsymbol{i}$ are roots of this polynomial. Note: we use the symbol $\boldsymbol{i}$ to represent the square root of negative one.

Name:______________________________________ Advanced 1999

School:__________ # 4

In the coordinate plane, a small circle is tangent to the x-axis, the y-axis, and the circle $x^2 + y^2 = 1$. Note: the large circle has radius one and center at the origin. See the figure at the right. Find the radius of the small circle. Find the exact radius and use your calculator to approximate this value to four decimal places.



Name:_______________________________________ Advanced 1999
School:___________ # 5

In each part below, the first four terms of a sequence are given. Assume that the given pattern continues. Write the next two terms and find a formula for the nth term. That is, you are given $\{a_1, a_2, a_3, a_4, \ldots\}$. You are to write a_5, a_6, and find a formula for a_n. For example, if you are given $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$, your answers would be

$a_5 = \frac{5}{6}$ $a_6 = \frac{6}{7}$ $a_n = \frac{n}{n+1}$.

(i) $\left\{\frac{1}{2}, \frac{4}{4}, \frac{7}{8}, \frac{10}{16}, \ldots\right\}$

$a_5 =$ $a_6 =$ $a_n =$

(ii) $\left\{\frac{1}{2\cdot 1}, \frac{4}{4\cdot 3}, \frac{9}{6\cdot 5}, \frac{16}{8\cdot 7}, \ldots\right\}$

$a_5 =$ $a_6 =$ $a_n =$

Hints for the 1999 Advanced test

#1 Suppose $x = r$ is a root of $f(x)$. Compute

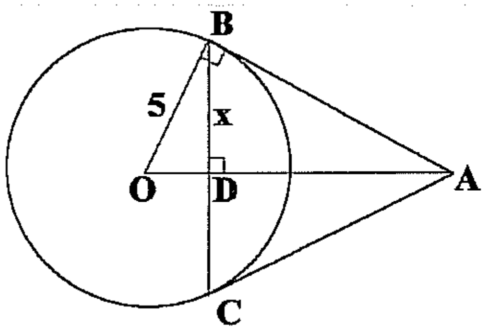
$$g\left(\frac{1}{r}\right) = \frac{c}{r^2} + \frac{b}{r} + a =$$

$$\frac{a\,r^2 + b\,r + c}{r^2} =$$

$$\frac{f(r)}{r^2} = \frac{0}{r^2} = 0$$

Hence, $1/r$ is a root of $g(x)$.
An alternate, but more difficult proof follows from the quadratic formula.

#2



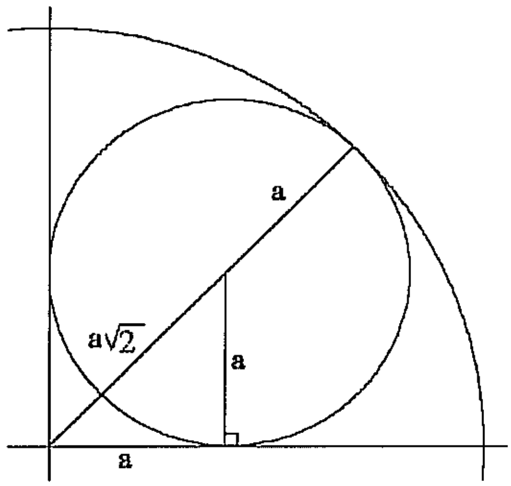
Let O be the center of the circle. Construct the line segments AO, OB, and BC. Let D be the point of intersection of AO and BC. The right triangles have angles 30° 60° and 90°. The length of segment BD = x satisfies:

$$x^2 + \left(\frac{5}{2}\right)^2 = 5^2 \quad or \quad x = \frac{5\sqrt{3}}{2}$$

It follows that the distance BC = 2x = $5\sqrt{3} \cong 8.6603$.

#3 If $x = 3 + i$ is to be a root, then $x = 3 - i$ must also be a root – in order for the coefficients to be real. The desired polynomial is: $(x-2)(x-3+i)(x-3-i)$ $= x^3 - 8x^2 + 22x - 20$.

#4



Let a denote the radius of the small circle. It follows that the distance from the center of the small circle to the origin is $a\sqrt{2}$. We have $a\sqrt{2} + a = 1$ or

$$a = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1 \cong 0.4142$$

#5

(i) $\left\{\frac{1}{2}, \frac{4}{4}, \frac{7}{8}, \frac{10}{16}, \ldots\right\}$

$$a_5 = \frac{13}{64} \quad a_6 = \frac{16}{128} \quad a_n = \frac{3n-2}{2^n}$$

(ii) $\left\{\frac{1}{2\cdot 1}, \frac{4}{4\cdot 3}, \frac{9}{6\cdot 5}, \frac{16}{8\cdot 7}, \ldots\right\}$

$$a_5 = \frac{25}{(10)\cdot 9} \quad a_6 = \frac{36}{(12)\cdot(11)}$$

$$a_n = \frac{n^2}{2n(2n-1)} = \frac{n}{2(2n-1)}$$

Hints and solutions for the 1999 Elementary test.

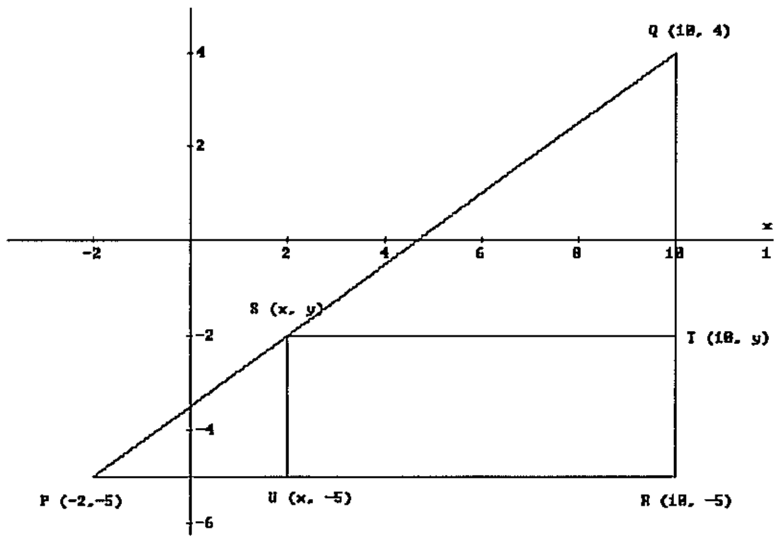
#1 The equation reduces to $1 - x = 5$. Or $x = -4$.

#2 Triangle ABC is similar to triangle BDC. Let x be the length of segment AC. It follows that $\frac{x}{8} = \frac{8}{4}$, or $x = 16$. The desired length of segment AD is

$x - 4 = 12$.

#3 The distance traveled by the center is equal to the arc subtended by a central angle of $\theta = 30° = \pi/6$ radians. This distance is $\pi/3$ cm. $\cong$ 1.0472 cm.

#4



Let S having coordinates (x, y) be the desired point. Draw construction lines as in the figure above. It follows that U (x, -5) will be one third of the way from P to R. Similarly, T (10, y) will be one third of the way from R to Q. We have

$$\frac{x+2}{10+2} = \frac{1}{3} \quad and \quad \frac{y+5}{4+5} = \frac{1}{3}$$

It follows that $(x, y) = (2, -2)$.

#5 (a)

$$\left(2 + \sqrt{-25}\right) - (3 + 4i) =$$

$$2 + 5i - 3 - 4i = -1 + i$$

(b)

$$\frac{15-10i}{2+i} = \left(\frac{15-10i}{2+i}\right) \cdot \left(\frac{2-i}{2-i}\right) =$$

$$\frac{30 - 15i - 20i + 10i^2}{5} = \frac{20-35i}{5} =$$

$$4 - 7i$$

Team Project – Hints and Solutions

(a) **Solution 1:** Triangle ADC is similar to triangle DBC. See Figure 1.

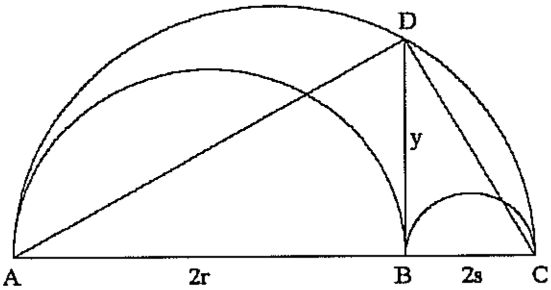


Figure 1

It follows that $\frac{y}{2r} = \frac{2s}{y}$. Solving for y gives $y = 2\sqrt{rs}$.

(a) **Solution 2:** Let E be the center of the large semi-circle. See Figure 2.

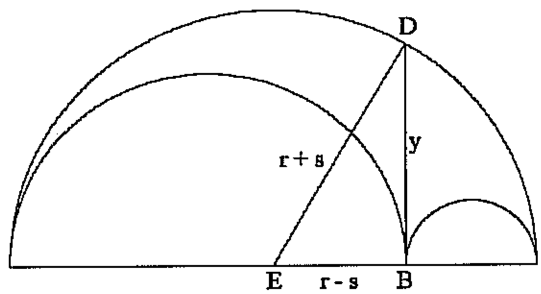


Figure 2

Draw construction line ED. It follows that

$$y^2 + (r-s)^2 = (r+s)^2.$$

Solving for y gives the same result as Solution 1.

(b) Let R be the center of the circle having radius r. Let S be the center of the circle having radius s. Then, connect R and S to the respective points of tangency. Construction line ST is parallel to PQ, making RST a right triangle with dimensions as shown in Figure 3.

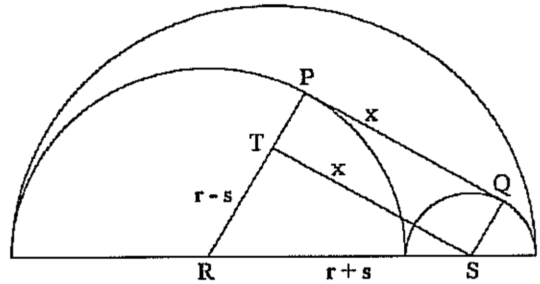


Figure 3

From the Pythagorean Theorem:

$$x^2 + (r-s)^2 = (r+s)^2$$

Solving for x = the length of PQ, we get $x = 2\sqrt{rs}$, which is the same as distance BD!

(c) The formulas are true when r = s.

Here is a problem for classroom discussion:

If we include the tangent line PQ in Figure 1, it appears that the lines AD and CD intersect the circles at the points of tangency P and Q respectively. **CAN**

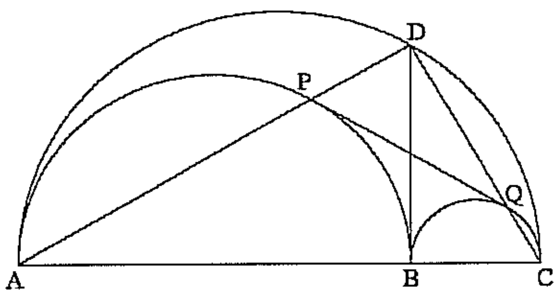


Figure 1 modified

YOU PROVE OR DISPROVE THIS?

Team Project 1998 School:______________________

Name 1:______________________ Name 2:________________________

Name 3:______________________ Name 4:________________________

Team Project – Instructions

1. Print school name and names of contestants neatly in the spaces provided above.

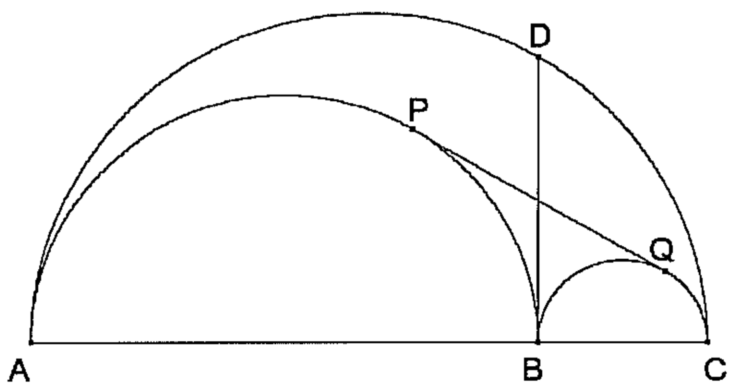
2. Use unlined paper for scratch work.

3. Write (or print) the complete solution, **including work required to get your answer**, neatly on the lined paper provided.

4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet as a cover sheet. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be written neatly, using correct English.

5. Good luck and good Mathematics!

Team Project – Problem



Three semi-circles have centers on the line segment AC, so that AB, AC, and BC are diameters of these semi-circles – as in the figure at the right. Point D, on the large semi-circle, is chosen so that the segment BD is perpendicular to segment AC. Line segment PQ is tangent to the two smaller circles, with P and Q as points of tangency. Let the length of segment AB be $2r$ and the length of segment BC be $2s$, with $r > s$.

(a) Find a formula for the length of segment BD in terms of r and s.
(b) Find a formula for the length of segment PQ in terms of r and s.
(c) Do your formulas still work when $r = s$?

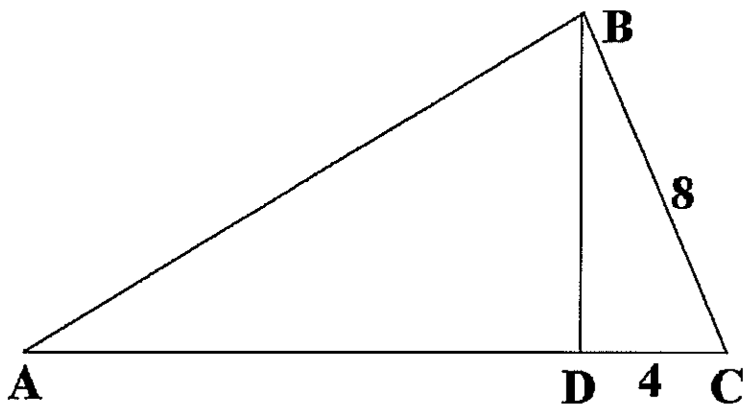
Name:____________________________________ Elementary 1999

School:__________ # 1

Solve for x: $$1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-x}}}=5$$

Name:________________________________ Elementary 1999
School:__________ # 2

In the figure, angle ABC and angle CDB are right angles. Segment BC has length 8 and segment CD has length 4. Find the length of segment AD.



Name:________________________________ Elementary 1999
School:__________ # 3

A cylinder having, diameter four centimeters, rolls through an angle of 30°. How far does the center of the cylinder travel as it rolls through this angle? Give the exact answer and the approximate answer rounded to four decimal places.

Name:____________________________________ Elementary 1999
School:__________ # 4

In the Cartesian coordinate plane, point P has coordinates (-2, -5) and point Q has coordinates (10, 4). Determine the coordinates of the point on line segment PQ, which is one third of the way from P to Q.

Name:____________________________________ Elementary 1999
School:____________ # 5

Simplify each expression and write it in the form $a + b\boldsymbol{i}$. Note: we use the symbol $\boldsymbol{i}$ to represent the square root of negative one.

(a) $\left(2+\sqrt{-25}\right)-(3+4i)$

(b) $\dfrac{15-10i}{2+i}$

A dietician is preparing a meal consisting of foods X, Y, and Z. Each ounce of food X contains 2 units of protein, 3 units of fat, and 4 units of carbohydrate. Each ounce of food Y contains 3 units of protein, 2 units of fat, and 1 unit of carbohydrate. Each ounce of food Z contains 3 units of protein, 3 units of fat, and 2 units of carbohydrate. If the meal must provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?

Name:________________________________ Elementary 2004 School:__________ # 2

The **Preposterous Pizza Place** sells a **Fear Factor Pizza**. Possible toppings are: anchovies, bacon, cauliflower, duck, eggplant, fish, garbanzos, and/or ham. You may have any number of these toppings, or no topping at all. How many possible **Fear Factor Pizzas** are there?

Name:______________________________ Elementary 2004 School:_________ # 3

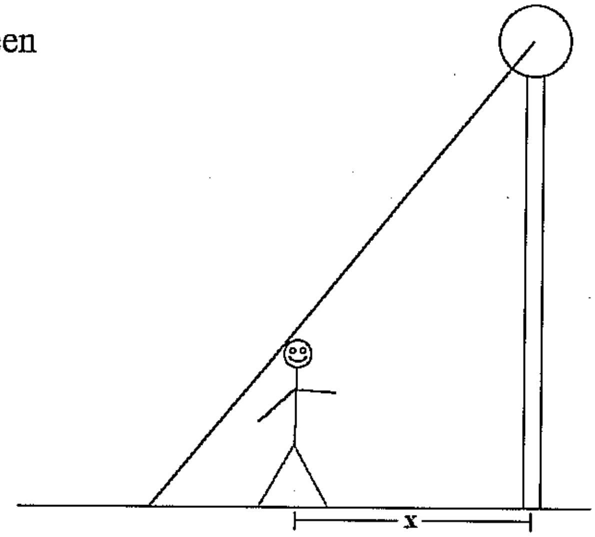
Divide $3x^{10} - 6x^9 + 4x^6 + 3x^5 - 9x^3 + 2x^2 + 16$ by $x - 2$. Find the quotient and remainder.

Name:________________________________ Elementary 2004 School:__________ # 4

Myrtle weeds the garden in an hour and a half. Ivy weeds the garden in 75 minutes. How long will it take to weed the garden if Myrtle and Ivy work together? Give your answer in minutes and seconds – rounded to the nearest second.

Name:______________________________ Elementary 2004 School:_________ # 5

A six-foot tall man is standing x feet away from a streetlight that is eighteen feet tall. Express the distance from the base of the streetlight to the tip of his shadow in terms of x.



Hints and Solutions for Elementary Test Problems 2004

#1 We let A denote the number of ounces of food X to be used in the meal; B the ounces of Y; and C the ounces of Z. We get the system of equations:

$$\begin{cases} 2A+3B+3C=25 \\ 3A+2B+3C=24 \\ 4A+B+2C=21 \end{cases}$$

Using standard methods, we may obtain the solutions:

$$A = 16/5 = 3.2 \text{ oz.}; \quad B = 21/5 = 4.2 \text{ oz.}; \quad C = 2 \text{ oz.}$$

#2 The number of possible pizzas is the number of subsets of a set containing 8 elements. It is $2^8 = 256$ pizzas.

#3 Either Synthetic Division or Long Division will give:

Quotient = $3x^9 + 4x^5 + 11x^4 + 22x^3 + 35x^2 + 72x + 144$ and Remainder = 304.

#4 Let A denote the area of the garden. The rates at which Myrtle and Ivy weed the garden are A/90 and A/75 (units of area per minute) respectively. If T is the time to weed the garden for Myrtle and Ivy working together, we have:

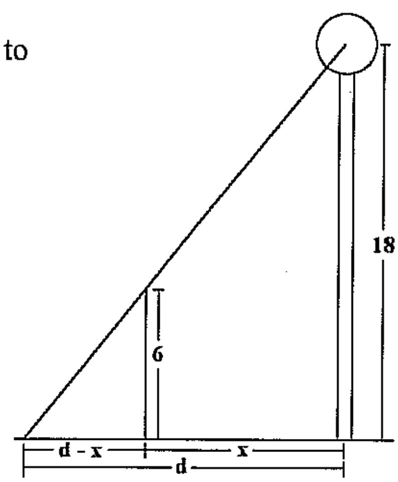
$$T\left(\frac{A}{90}+\frac{A}{75}\right)=A \quad \text{or} \quad T\left(\frac{1}{90}+\frac{1}{75}\right)=1.$$ Solving for T gives

$$T=\frac{450}{11}=40.909090\cdots \text{ min.}$$

or $T \approx$ 40 min. 55 sec.

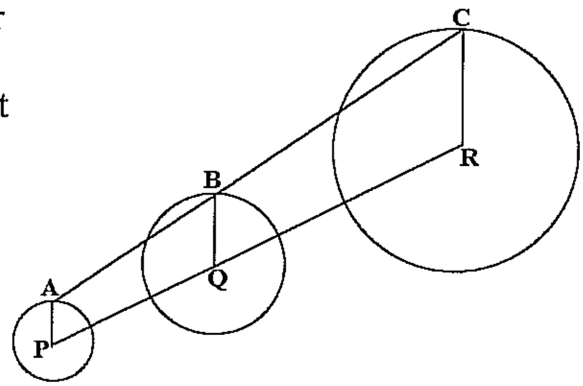
#5 We let d denote the distance from the base of the streetlight to the end of the man's shadow. By similar triangles, we have:

$$\frac{d-x}{6}=\frac{d}{18}.$$ Solving for d gives $d=\frac{3x}{2}$.



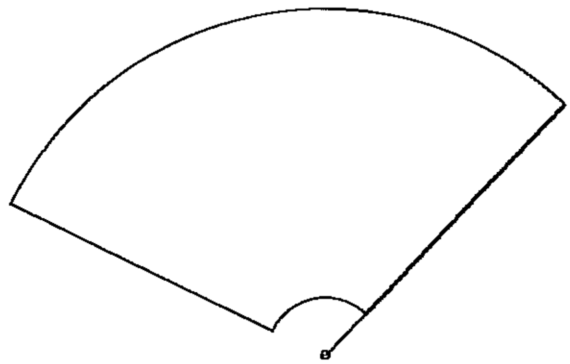
Name:________________________________ Advanced 2004 School:_________ # 1

The day after a lunar eclipse, Farmer Brown discovered three crop circles in his corn field. (See the figure at the right – not necessarily drawn to scale.) With the help of his farm hand, Farmer Brown began to document this phenomenon. There were strange marks at points A, B, and C, one on each circle. Points A, B, and C were collinear. The centers of the circles, denoted by P, Q, and R, were also collinear. Farmer Brown measured the radii PA and QB to be 15 yards and 40 yards, respectively. He measured the distances PQ and QR to be 100 yards and 150 yards, respectively. He also discovered that the lines PA, QB, and RC were parallel. It was chore time and he didn't get the radius RC measured. Find distance RC.



Name:________________________________ Advanced 2004 School:__________ # 2

The wiper blade on Jan's car is 16 inches long and wipes an area bounded by concentric, circular arcs and lines passing through the center of these arcs. The large circular arc has radius 20 inches and the blade sweeps through an angle of $130°$ on a flat window. Find the area covered by this wiper blade to the nearest hundredth of a square inch.



Name:________________________________ Advanced 2004 School:__________ # 3

Find all solutions to the system: $\begin{cases} xy = 3x + 4 \\ x^2 + y^2 = 6y + 1 \end{cases}$

Name:______________________________ Advanced 2004 School:__________ # 4

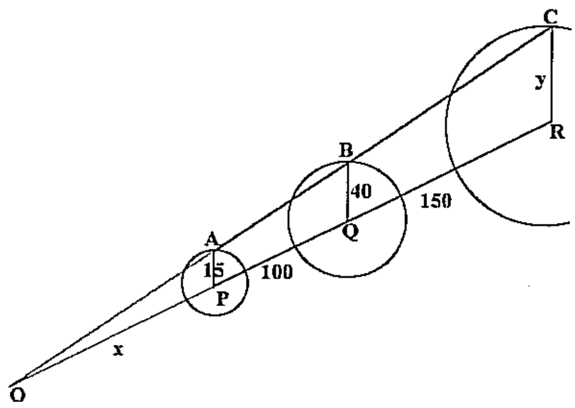
Solve for θ. Find all exact solutions in radians, with $0 \le \theta < 2\pi$.

$$\sin(4\theta) + \cos(2\theta) = 0$$

The partial fraction decomposition for the rational expression $\frac{5x^2+4x+2}{x^3+x}$ is $\frac{2}{x}+\frac{3x+4}{x^2+1}$. Find the partial fraction decomposition for the following rational expression: $\frac{7x^2+3x-5}{x^3-x^2+4x-4}$

Hints and solutions for Advanced Test Problems 2004

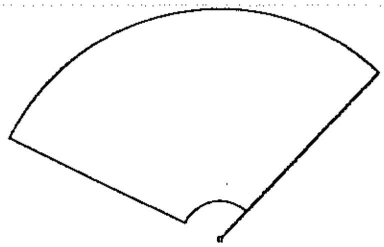
#1 Extend segments AC and PR until they meet at O. Triangles OPA, OQB, and ORC are similar. Letting $x = OP$ and $y = RC$, it follows that:



$$\frac{x}{15} = \frac{x+100}{40} = \frac{x+250}{y}.$$

Solving the first equation for x gives $x = 60$ yds. Substituting this value in the second equation and solving for y gives (the radius RC) $y = 77.5$ yds.

#2 We have two circular sectors – the large one with radius 20 inches and the small one with radius 4 inches. We remember that the area of a circular sector is



$$A = \frac{1}{2}r^2\theta$$

where r is the radius of the sector and θ is the central angle (in radians) of the sector. In this case, the central angle is $\theta = \frac{13\pi}{18}$. Subtracting the area of the small sector from the area of the large sector gives the desired area. $\frac{1}{2}\left(20^2 - 4^2\right)\frac{13\pi}{18} = \frac{416\pi}{3} \cong 435.63\,\text{in}^2$

#3 Solving the first equation for y gives: $y = \frac{3x+4}{x}$. (*)

Substituting in the second equation, clearing fractions, and simplifying gives the equation:

$x^4 - 10x^2 + 16 = 0$, which has solutions $x = \pm\sqrt{2}$ $x = \pm 2\sqrt{2}$.

Putting each of these four values back in (*), gives the four points:

$$\left(\sqrt{2}, 3+2\sqrt{2}\right),\ \left(-\sqrt{2}, 3-2\sqrt{2}\right),\ \left(2\sqrt{2}, 3+\sqrt{2}\right),\ \left(-2\sqrt{2}, 3-\sqrt{2}\right)$$

Hints and solutions for Advanced Test Problems 2004

#4 Making the substitution $\varphi = 2\theta$, the equation becomes $\sin(2\varphi) + \cos\varphi = 0$, where we are now looking for solutions satisfying $0 \le \varphi < 4\pi$. Using a double angle formula and factoring gives $\cos\varphi(2\sin\varphi + 1) = 0$.

Setting the first factor equal to zero gives: $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

Setting the second factor equal to zero gives: $\varphi = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$.

Dividing by two gives the values:

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

#5 We factor the denominator: $x^3 - x^2 + 4x - 4 = (x-1)(x^2+4)$

The partial fraction decomposition is of the form:

$\frac{7x^2+3x-5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$. Clearing fractions and simplifying gives:

$7x^2 + 3x - 5 = (A+B)x^2 + (C-B)x + (4A-C)$. Equating coefficients gives the three equations: $\begin{cases} A+B=7 \\ C-B=3 \\ 4A-C=-5 \end{cases}$, whose solution is:

(A, B, C) = (1, 6, 9). Thus, the partial fraction decomposition is:

$$\frac{1}{x-1} + \frac{6x+9}{x^2+4}.$$

Team Project 2004 School:____________________

Name 1:____________________ Name 2:____________________

Name 3:____________________ Name 4:____________________

Team Project - Instructions

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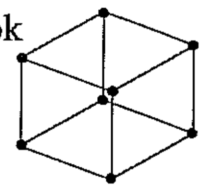
4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet as a cover sheet
5. Good luck and good Mathematics!

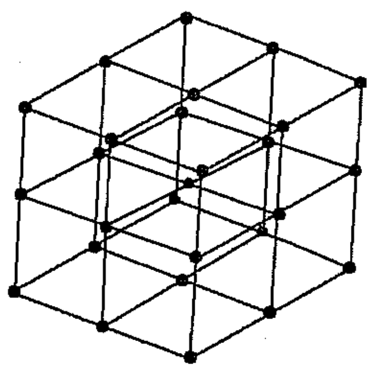
Team Project

Alice is telling Betty about her attempt to get into the Guinness Book of World Records.



1 x 1 x 1

Alice: "I am making a big, cubic lattice from toothpicks and I need to know how many toothpicks to buy."
Betty: "Will you hold them together with gumdrops?"
Alice: "Maybe – but I thought drops of glue might be cheaper, sturdier, and I wouldn't be tempted to eat it. Here, I made a **one by one by one** (1 x 1 x 1) cubic lattice and a **two by two by two** (2 x 2 x 2) cubic lattice."
Betty: "I see. The 1 x 1 x 1 lattice took 12 toothpicks and the 2 x 2 x 2 lattice took 54 toothpicks. I suppose you want a formula for the number of toothpicks needed to make an n x n x n lattice."
Alice: "That's the idea."



2 x 2 x 2

Your task is to answer the following:

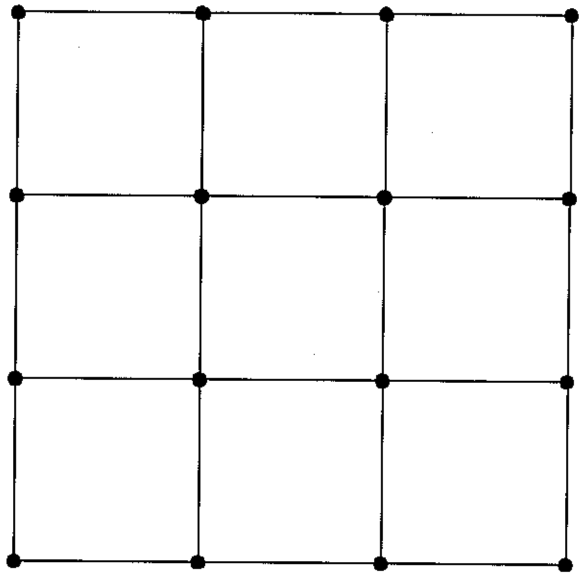
(a) How many toothpicks will be needed for a 3 x 3 x 3 lattice?

(b) Find a formula for the number of toothpicks needed for an n x n x n lattice. Also, give a reasonable justification as to why your formula is correct.

Hints and solution to Team Project 2004

(a) How many toothpicks will be needed for a 3 x 3 x 3 lattice?

Orient the lattice so the edges are aligned north-south, east-west, and perpendicular to those directions. We count the number of east-west toothpicks in one layer to be 12. There are 4 layers. So, there are 48 east-west toothpicks. Multiplying by 3, for the three different directions, gives 144 toothpicks in the lattice.



One layer of a 3 x 3 x 3 lattice

(b) Find a formula for the number of toothpicks needed for an n x n x n lattice. Also, give a reasonable justification as to why your formula is correct.

The formula for the n x n x n case is obtained in a way similar to the 3 x 3 x 3 case. Orient the lattice so the edges are aligned north-south, east-west, and perpendicular to those directions. We count the number of east-west toothpicks in one layer to be

$n(n + 1)$. There are $(n + 1)$ layers. So, there are $n(n+1)^2$ east-west toothpicks.

Multiplying by 3, gives $3n(n+1)^2$ toothpicks in the lattice.

Name:__________________________________ Advanced 2007

School:__________________________________ # 1

We let $\log(r) = \log_{10}(r)$ = the logarithm to the base ten of r. Simplify the following expression.

$$\log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \log\left(\frac{4}{5}\right) + \cdots + \log\left(\frac{198}{199}\right) + \log\left(\frac{199}{200}\right)$$

Name:______________________________ Advanced 2007

School:______________________________ # 2

Find the exact values of k and $\tan\varphi$ so that:

$$3\sin\alpha + 5\cos\alpha = k\sin(\alpha + \varphi)$$

for all angles .

Name:______________________________ Advanced 2007

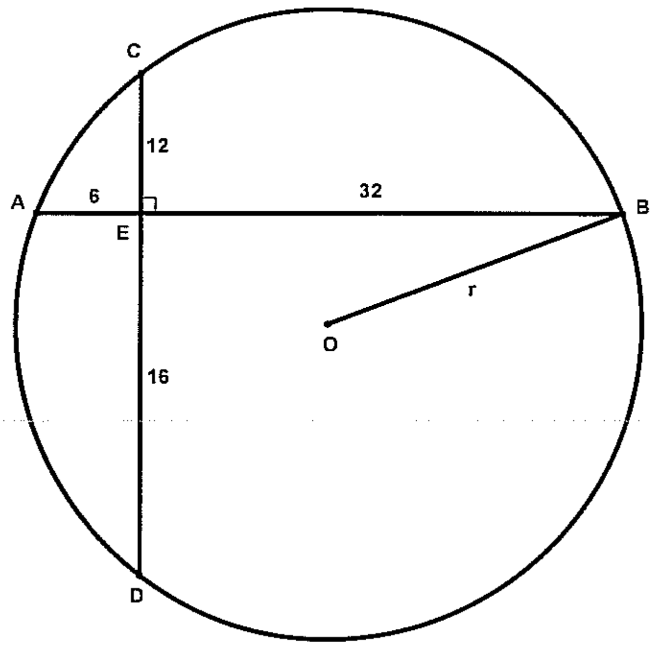
School:______________________________ # 3

Two sides of a triangle have lengths of 3 cm and 7 cm, and meet at an angle of 60°. Find the exact area of the triangle.

Name:________________________________ Advanced 2007

School:________________________________ # 4

Two chords, $\overline{AB}$ and $\overline{CD}$ of a circle having center O are perpendicular at point E, with AE = 6, EB = 32, CE = 12, and ED = 16. (The illustration at the right is not necessarily drawn to scale.)



Find the radius r of circle O.

Name:______________________________ Advanced 2007

School:______________________________ # 5

The **Three Men and a Ladder** remodeling company sent Larry, Moe, and Curly to paint your house. Larry works 1.5 times as fast as Moe. Curly works twice as fast as Moe. Together the three men paint your house in eight hours. How long would it take Larry to paint your house, if he had to do it by himself?

Hints and Solutions for the Advanced Contest

#1 Using properties of the logarithm, we can convert the sum into the log of a product:

$$\log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \log\left(\frac{4}{5}\right) + \cdots + \log\left(\frac{198}{199}\right) + \log\left(\frac{199}{200}\right) =$$

$$\log\left(\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{198}{199}\cdot\frac{199}{200}\right) = \log\left(\frac{2}{200}\right) = \log\left(\frac{1}{100}\right) = -2$$

#2 We remember the identity:

(*) $\sin(\alpha+\varphi) = \cos\varphi\sin\alpha + \sin\varphi\cos\alpha$. Since we want

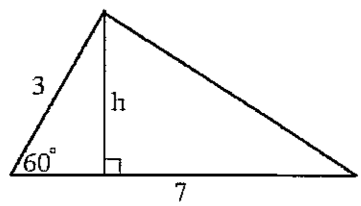
$\sin(\alpha+\varphi) = \frac{3}{k}\sin\alpha + \frac{5}{k}\cos\alpha$, we may reconcile these two equations by letting

$\cos\varphi = \frac{3}{k}$ *and* $\sin\varphi = \frac{5}{k}$. This implies that $\left(\frac{3}{k}\right)^2 + \left(\frac{5}{k}\right)^2 = 1$ and

$k = \sqrt{34}$ *or* $k = -\sqrt{34}$. Finally, we have $\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{5}{3}$.

#3 We draw a rough sketch of the triangle. The area is

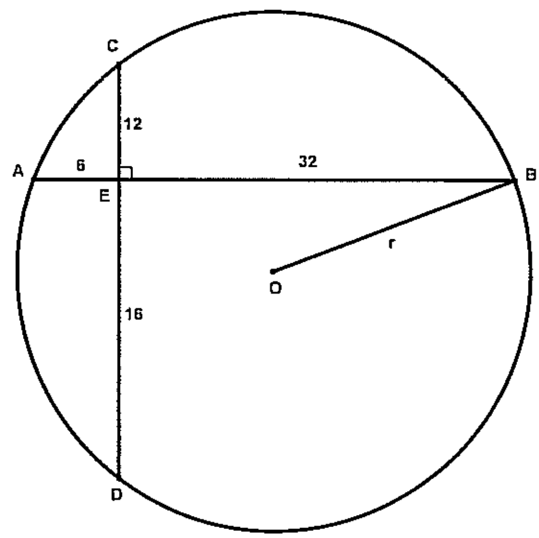
$$A = \frac{1}{2}(base)\cdot h = \frac{1}{2}\cdot 7\cdot 3\sin(60^o) = \frac{21\sqrt{3}}{4}$$



#4 We may set up a coordinate system with EB as the positive x-axis and EC as the positive y-axis. The center of the circle will lie on the perpendicular bisector of chord AB and also on the perpendicular bisector of chord CD. Thus, the coordinates of the center are:

(13, – 2). It is a simple matter to find the radius.

$$r = \sqrt{365}$$



#5 Let M = the rate at which Moe paints (in house per hour.) We have

$$(1.5M + M + 2M)\cdot 8 = 1, \text{ or}$$

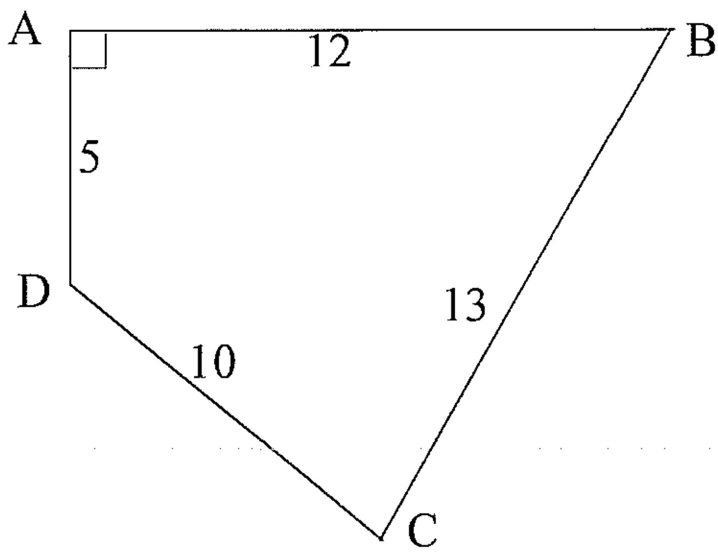
$$M = \frac{1}{36}.$$

It would take Larry 24 hours to paint your house.

Name:______________________________ Elementary 2007

School:______________________________ # 1

The quadrilateral ABCD has sides AB = 12, BC = 13, CD = 10, and DA = 5; with a right angle at vertex A. Find the area of ABCD.



Name:________________________________ Elementary 2007

School:________________________________ # 2

My coin jar contains only dimes, nickels, and pennies. Thirty-three of the coins are dimes, twenty-five percent of the coins are nickels, and 4/9 of the coins are pennies. How much money is in my coin jar?

Name:________________________________ Elementary 2007

School:________________________________ # 3

What is the greatest prime factor of $55^{100} + 55^{101} + 55^{102}$?

Name:________________________________ Elementary 2007

School:_______________________________ # 4

In each part below, you are given the first five terms of a sequence $\{a_n\}$ starting with n = 1. Assume that the pattern continues. Write the next two terms and find a formula for the nth term. For example, if you are given $\left\{\frac{1}{2},\frac{-2}{3},\frac{3}{4},\frac{-4}{5},\frac{5}{6},\ldots\right\}$, your answers could be:

$a_6 = \frac{-6}{7}$, $a_7 = \frac{7}{8}$, $a_n = \frac{(-1)^{n+1}n}{n+1}$.

(i) $\left\{\frac{1}{2},\frac{2}{5},\frac{3}{8},\frac{4}{11},\frac{5}{14},\ldots\right\}$

$a_6 =$ $a_7 =$ $a_n =$

(ii) $\left\{\frac{2}{9-4},\frac{4}{9-16},\frac{8}{25-16},\frac{16}{25-36},\frac{32}{49-36},\ldots\right\}$

$a_6 =$ $a_7 =$ $a_n =$

Name:________________________________ Elementary 2007

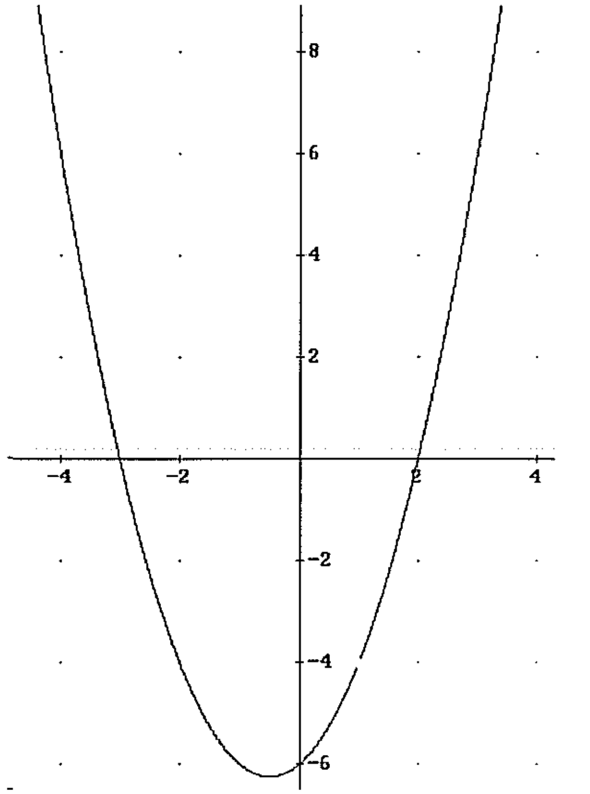
School:________________________________ # 5

Jack is confused. "The graph of a rational function should have a vertical asymptote at each x-value that makes the denominator zero. However, when I plot the graph of

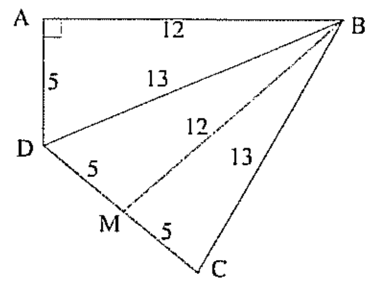
$$y = \frac{x^3 - 7x + 6}{x - 1}$$

with my CAS, there is no vertical asymptote at x = 1. The graph looks like a parabola. How can this be?"

Write a few sentences explaining why this "phenomenon" happened.



Hints and Solutions for the Elementary Contest



#1 We draw the construction line BD. By the Pythagorean Theorem, BD = 13. (Note: the figure may not be drawn to scale.) We let M be the midpoint of side CD and draw construction line BM. We determine the length BM = 12. The quadrilateral is composed of three congruent triangles, each having area 30 square units. The area of ABCD is 90 square units.

#2 Let D = the number of dimes, N = the number of nickels, and P = the number of pennies. We have: D = 33 and the equations

$$\begin{cases} (33+N+P)\cdot 0.25 = N \\ (33+N+P)\cdot \dfrac{4}{9} = P \end{cases}$$

Solving, gives N = 27 and P = 48. So, the amount of money in the jar is:
10 D + 5 N + P = 330 + 135 + 48 = 513 cents or $5.13.

#3 We factor 55^{100} from the expression to get:

$$55^{100} + 55^{101} + 55^{102} = (1 + 55 + 3025)\cdot 55^{100}$$
$$= (3081)\cdot 5^{100}\cdot 11^{100} = 1027\cdot 3\cdot 5^{100}\cdot 11^{100}$$
$$= 13\cdot 79\cdot 3\cdot 5^{100}\cdot 11^{100}$$

Hence, 79 is the greatest prime factor.

#4 For (i) $\left\{\frac{1}{2},\frac{2}{5},\frac{3}{8},\frac{4}{11},\frac{5}{14},\cdots\right\}$, we have: $a_6 = \frac{6}{17}, a_7 = \frac{7}{20}, \cdots a_n = \frac{n}{3n-1}$.

For (ii) $\left\{\frac{2}{9-4},\frac{4}{9-16},\frac{8}{25-16},\frac{16}{25-36},\frac{32}{49-36},\cdots\right\}$, we may write the sequence:

$\left\{\frac{2}{3^2-2^2},\frac{-4}{4^2-3^2},\frac{8}{5^2-4^2},\frac{-16}{6^2-5^2},\frac{32}{7^2-6^2},\cdots\right\}$. We have:

$$a_6 = \frac{-64}{8^2-7^2} = \frac{-2^6}{(8-7)(8+7)} = \frac{-2^6}{15},\ a_7 = \frac{2^7}{17}, \cdots a_n = \frac{(-1)^{n+1}2^n}{2n+3}$$

#5 The numerator of the rational function: $y = \frac{x^3 - 7x + 6}{x-1}$ is divisible by (x – 1). That is, $y = \frac{x^3-7x+6}{x-1} = \frac{(x-1)(x^2+x-6)}{(x-1)}$, which is equal to $y = x^2 + x - 6$ for x not equal to one. So, the graph of the rational function is a parabola – with the point (1, – 4) missing.

Team Project 2007 School:______________________________

Name 1:______________________ Name 2:______________________

Name 3:______________________ Name 4:______________________

Team Project Instructions

1. **Print** school name and names of contestants neatly in the spaces provided above.
2. Use unlined paper for scratch work.
3. Write (or print) the complete solution, **including work required to get your answer**, neatly on the lined paper provided. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be written neatly, using correct English.
4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet clipped to it as a cover sheet.
5. Good luck and good Mathematics!

Read the conversation between Joe and Jane on this page and the next and then do the Math Team Project.

Joe: "I am remembering the spirit of Archimedes by reproducing one of his great achievements."

Jane: "Will you jump from your bathtub and run through the streets yelling 'EUREKA'?"

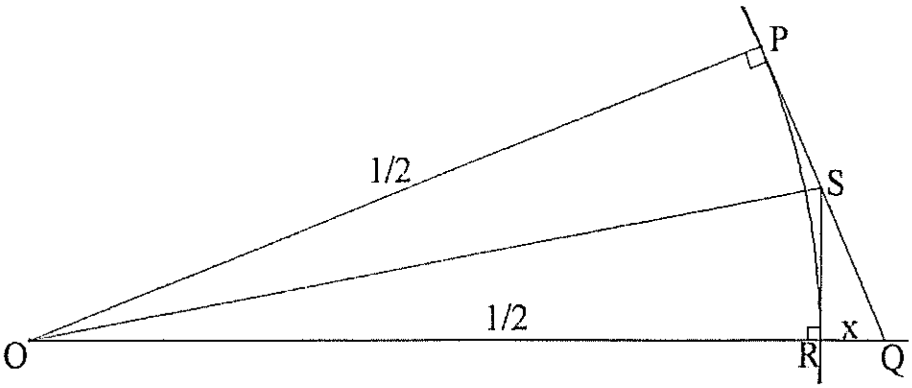
Joe: "No, I want to recreate Archimedes' approximation to . He began with a circle having diameter 1 (one) and created two sets of regular polygons (equal sides and equal interior angles). For one set (the circumscribed polygons), he started with a square, having sides of length 1 (one) that were tangent to the circle at the midpoint of each side. His next polygon was a regular octagon with sides tangent to the circle at the midpoint of each side."

Jane: "So, he got polygons having 4, 8, 16, 32, ... sides. As the number of sides became larger, the perimeter of the polygon approached the circumference of the circle – which is ."

Joe: "I found a **recursion formula** for b_n = the length of one side of the n^{th} circumscribed polygon."

Jane: "OK, you start with $b_1 = 1$, calculate b_2 from b_1, calculate b_3 from b_2, etc. It is a simple matter to find the perimeter of each polygon by multiplying the length of a side by the number of sides. How did you determine the recursion formula?"

Joe: "The figure shows part of a circle with center O having radius ½ and parts of the circumscribed polygons having side lengths b_n and b_{n-1}. In the figure,



we let $x = QR$, $PQ = \frac{b_{n-1}}{2}$ and $RS = \frac{b_n}{2}$. From the Pythagorean Theorem (applied to OPQ) we have: $\left(x+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{b_{n-1}}{2}\right)^2$ or $2x = \sqrt{(b_{n-1})^2+1} - 1$. Now OPQ is similar to SRQ and $\frac{QR}{RS} = \frac{QP}{PO}$ giving $b_n = \frac{2x}{b_{n-1}} = \frac{\sqrt{(b_{n-1})^2+1}-1}{b_{n-1}}$."

Jane: "Neat! The operations required to calculate b_n from b_{n-1} are operations that Archimedes could have performed by hand."

Joe: "Of course, I used my calculator to do the calculations. See, I made a table of the first few approximations to using circumscribed polygons."

n	b_n	number of sides	perimeter
1	1.000000000	4	4.000000000
2	0.414213562	8	3.313708499
3	0.198912367	16	3.182597878
4	0.098491402	32	3.151724906
5	0.049126848	64	3.144118376

Math Team Project

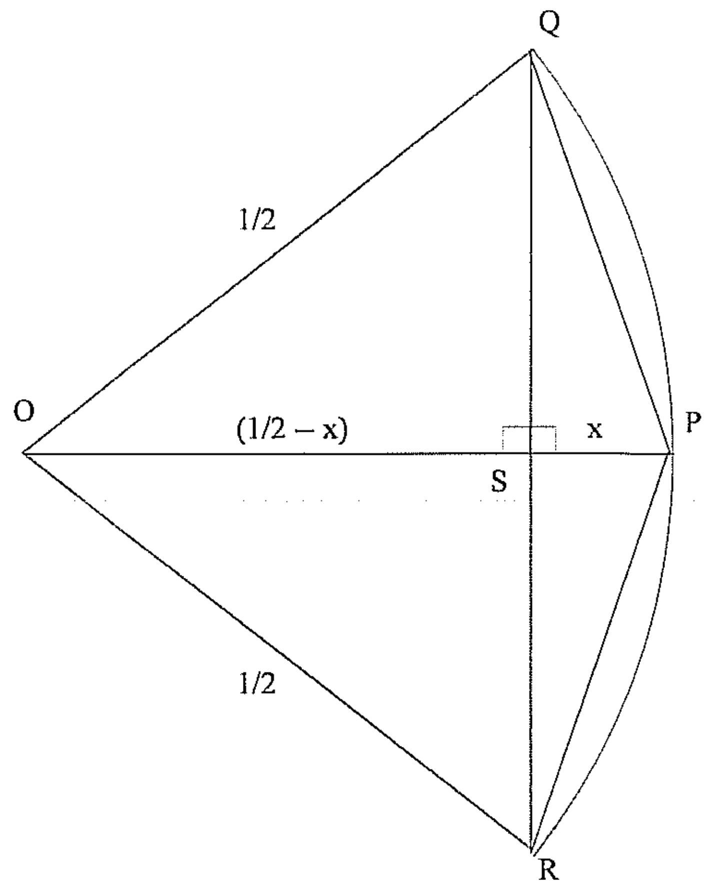
Perform similar work for the regular inscribed polygons having 2^{n+1} sides with vertices on the circle of diameter 1 (one.) In particular:

(a) Begin with a square having sides of length $c_1 = \frac{\sqrt{2}}{2}$.

(b) Letting c_n denote the length of one side of the n^{th} inscribed polygon, determine a recursion formula for c_n in terms of c_{n-1}.

(c) Calculate the perimeters of the first five inscribed polygons.

Team Project 2007   Hints and Solutions



The figure shows part of a circle with center O having radius ½ and parts of the inscribed polygons having side lengths $PQ = c_n$ and $QR = c_{n-1}$. In the figure, we let $x = SP$. Applying the Pythagorean theorem to triangle OSQ, we get

$$\left(\frac{1}{2} - x\right)^2 + \left(\frac{c_{n-1}}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

or

$$x = \frac{1 - \sqrt{1 - (c_{n-1})^2}}{2}.$$

Applying the Pythagorean theorem to triangle PSQ, we get

$$c_n = \sqrt{x^2 + \left(\frac{c_{n-1}}{2}\right)^2}$$

replacing x gives the recursion relation:

$$c_n = \sqrt{\frac{1 - \sqrt{1 - (c_{n-1})^2}}{2}}.$$

Below is a table of the first five approximations to using inscribed polygons.

n	**c_n**	**Number of sides**	**Perimeter**
1	0.7071067811	4	2.828427124
2	0.3826834323	8	3.061467458
3	0.1950903220	16	3.121445152
4	0.09801714032	32	3.136548490
5	0.04906767432	64	3.140331157