Name: ______________________________ Advanced 1998 # 1

School: ____________________________

Jar A contains one red ball and three green balls. Jar B contains six red balls and two green balls. The following experiment is performed:

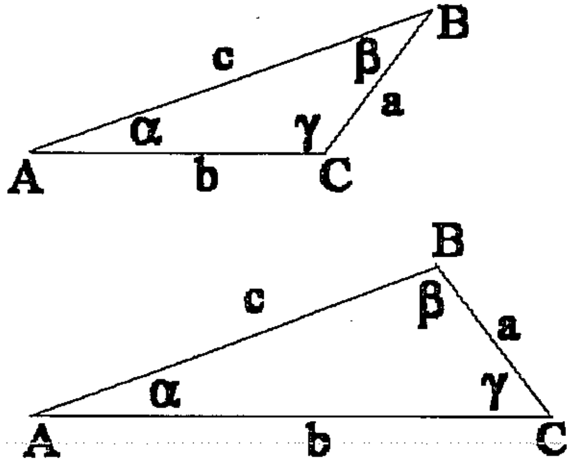
> A fair (six sided) die is rolled. If the die shows 1 or 2, a ball is randomly drawn from jar A. If the die shows 3, 4, 5, or 6, a ball is randomly drawn from jar B.

What is the probability that a green ball is drawn?

Name:____________________ Advanced 1998 # 2

School:____________________

Triangle ABC is given the standard labeling. Suppose the measure of angle α is $20°$, side c has length 30, and side a has length 15. There are two such triangles possible (as in the figure.) Find the two possible values for angle γ (opposite side c.) Give your answer in degrees, approximated to four decimal places.



Name: ______________________________ Advanced 1998 # 3

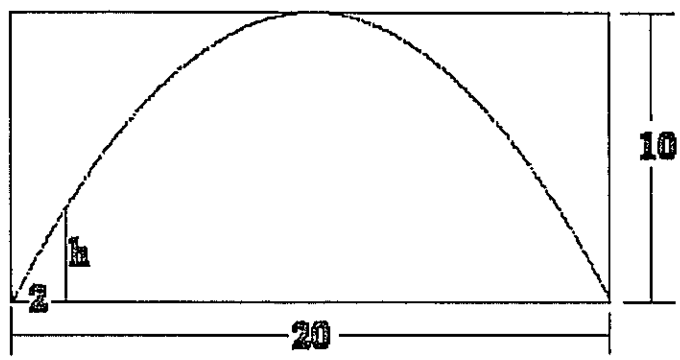
School: ____________________________

A square of perimeter 20 is inscribed in a square of perimeter 28. What is the greatest possible distance between a vertex of the inner square and a vertex of the outer square?

Name: ______________________________ Advanced 1998 # 4

School: ____________________________

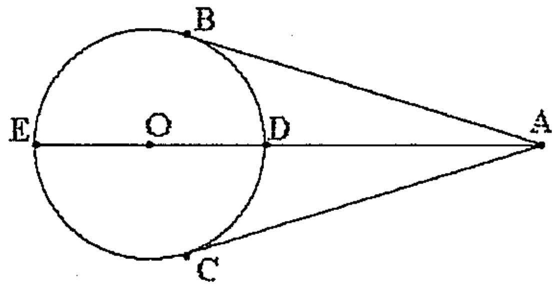
A horizontal bridge is in the shape of a parabolic arch. The dimensions given in the figure are in meters. Find the height, h, of the arch two meters from the end.



Name: ______________________________ Advanced 1998 # 5

School: ____________________________

In the figure, line segments AB and AC are tangent to the circle. Points B and C are points of tangency. Segment AB has length seven units and segment AD has length three units. Find the length of segment DE.



Name:______________________________ Advanced 1998 # 6

School:____________________________

Find all real solutions to the equation:

$$(x^2 - x - 1)^{(x^2-10x+24)} = 1$$

Name: ______________________________ Elementary 1998 # 1

School: ______________________________

Solve the equation. Find all solutions, real and complex

$$2x^3 + x^2 + 8x + 4 = 0$$

Name:________________________________ Elementary 1998 # 2

School:______________________________

The will of an eccentric millionaire reads as follows:

> "I leave 4/17 of my estate to my daughter, 7/13 of the remainder to my husband, 2/3 of the remainder to my son, and the remaining $8,000,000 to my cat."

What is the total amount of the estate?

Name: ________________________________ Elementary 1998 # 3

School: ________________________________

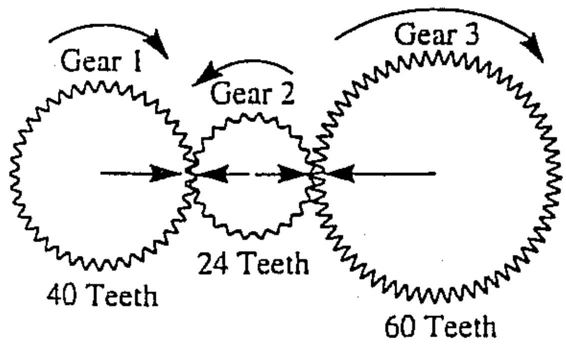
Solve the system of equations.

$$\begin{cases} x^2 + y^2 = 5 \\ x + 2y = 3 \end{cases}$$

Name: ______________________________ Elementary 1998 # 4

School: ______________________________

Determine how many complete revolutions each gear must make before the arrows are lined up again.



Revolutions of Gear 1: __________

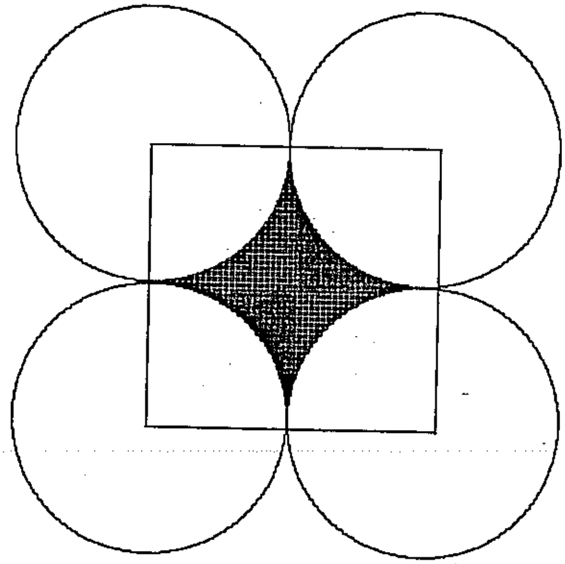
Revolutions of Gear 2: __________

Revolutions of Gear 3: __________

Name:______________________________________ Elementary 1998 # 5

School:______________________________________

Four circles of radius five feet are tangent to each other as in the figure. Their centers form the vertices of a square. What is the area of the shaded region which is both inside the square and outside all four circles?



Name: ______________________________

School: ______________________________

On a rainy day in April, ***Umber's Umbrella Store*** sold fifty umbrellas. They sold five times as many red umbrellas as blue umbrellas. They sold three times as many green umbrellas as blue umbrellas. They sold nine more black umbrellas than blue umbrellas. They sold seven fewer purple umbrellas than black umbrellas. They sold five purple umbrellas. All the rest of the umbrellas they sold were white. How many of each color were sold?

Red umbrellas sold: ________
Blue umbrellas sold: ________
Green umbrellas sold: ________
Black umbrellas sold: ________
Purple umbrellas sold: 5
White umbrellas sold: ________

Hints and Solutions, Elementary Mathematics Competition - 1998

1. We may factor the equation by grouping:

$$(2x^3 + x^2) + (8x + 4) =$$
$$(2x + 1)x^2 + (2x + 1)4 =$$
$$(2x + 1)(x^2 + 4) = 0$$

The solutions are $x = -1/2, 2i, -2i$.

2. Let E denote the amount of the estate. The daughter's share is 4E/17 with a remainder of 13E/17. The husband's share is 7E/17 with a remainder of 6E/17. The son's share is 4E/17 with a remainder of 2E/17 – which goes to the cat. Hence,

E = $68,000,000.

3. For the system of equations.

$$\begin{cases} x^2 + y^2 = 5 \\ x + 2y = 3 \end{cases}$$

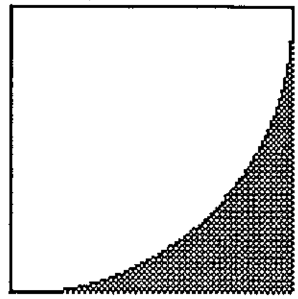
we solve the second equation for x, giving $x = 3 - 2y$. Substituting this x in the second equation yields
$5y^2 - 12y + 4 = (5y - 2)(y - 2) = 0$.
The solutions are:
$(x, y) = (-1, 2)$ and $(x, y) = (11/5, 2/5)$.

4. Let R_1 denote the revolutions of Gear 1; R_2 denote the revolutions of Gear 2; and R_3 denote the revolutions of Gear 3. We fix a point near each gear and count the number of teeth T (the same for each gear) which pass that point until the arrows are again aligned. It follows that

$$T = 40 \cdot R_1 = 24 \cdot R_2 = 60 \cdot R_3.$$

The smallest possible T will correspond to the least common multiple (LCM) of 40, 24, and 60. In symbols,
$T = \mathrm{LCM}(40, 24, 60) =$
$\mathrm{LCM}(2^3 \cdot 5, 2^3 \cdot 3, 2^2 \cdot 3 \cdot 5) = 2^3 \cdot 3 \cdot 5$.
Hence, $R_1 = 3$, $R_2 = 5$, and $R_3 = 2$.

5. We may cut the square into four equal squares, which look like the figure at the right. The shaded area in each of these smaller squares is:



$25 - 25\pi/4$.

It follows that the shaded area in the large square is: $100 - 25\pi$.

6. We let R = the number of Red, B = the number of Blue, G = the number of Green, K = the number of blacK, P = the number of Purple, and W = the number of White umbrellas sold. We have:

$R = 5B$
$G = 3B$
$K = B + 9$
$P = K - 7$
$P = 5$
$W = 50 - R - B - G - K - P$.

Solving this system gives:

$R = 15$
$B = 3$
$G = 9$
$K = 12$
$P = 5$
$W = 6$.

Hints and Solutions, Advanced Mathematics Competition - 1998

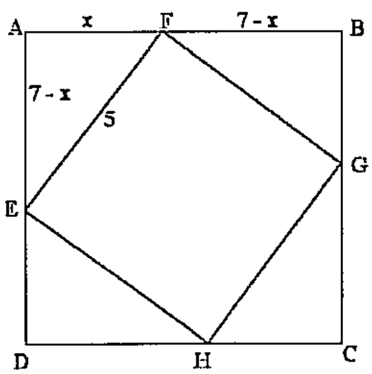
1. The probability of the die: (i) showing 1 or 2 is 1/3; (ii) showing 3, 4, 5, or 6 is 2/3. The probability of the die: (i) showing 1 or 2 ***followed by drawing a green ball*** is (1/3)(3/4); (ii) showing 3, 4, 5, or 6 ***followed by drawing a green ball*** is (2/3)(1/4). Adding, gives the probability of drawing a green ball = 5/12.

2. The Law of Sines gives:

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a}$$

or $\sin\gamma = 2\sin(20°)$
which has two solutions ($0° < \gamma < 180°$):
$\gamma = \arcsin(2\sin(20°)) \cong 43.1602°$
$\gamma = 180° - \arcsin(2\sin(20°)) \cong 136.8398°$

3. We draw and label a figure as below



From the Pythagorean Theorem applied to triangle AFE we have

$$x^2 + (7 - x)^2 = 25,$$

which has solutions x = 3, 4. Using x = 3, the greatest possible distance between a vertex of the inner square and a vertex of the outer square is the length of the segment FC. This length is

$$FC = \sqrt{65}$$

which may be obtained by a second application of the Pythagorean Theorem to the triangle FBC.

4. Set up a coordinate system with x-axis along the bottom of the parabolic arch and origin at the left-most point on the parabola (y-axis pointing up.) So, the parabola must pass through the points

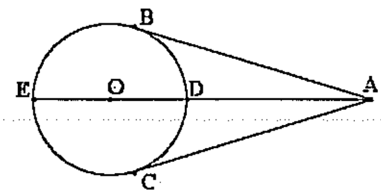
(0, 0), (10, 10), and (20, 0).

All parabolas passing through the first and last points must be of the form

$$y = A\,x(x - 20).$$

Imposing the condition (x, y) = (10, 10), determines A = -1/10. By setting x = 2, we get y = h = 3.6 meters.

5. We let r denote the radius of the circle. Then, segment OB has length r and segment OA has length r + 3. Applying the Pythagorean Theorem to triangle ABO (right angle at B), we get



$$49 + r^2 = (r + 3)^2$$

Solving gives r = 20/3. So that the length of segment DE = 2r = 40/3.

6. An equation of the form

$B^E = 1$ will be true in case:
(i) $B = 1$, (ii) $B = -1$ and E is an even integer, or (iii) $E = 0$ and B is not zero.
For our equation

$$(x^2 - x - 1)^{(x^2 - 10x + 24)} = 1$$

Condition (i) gives:
$x^2 - x - 1 = 1$, having solutions x = -1, 2.
The first part of condition (ii) gives:
$x^2 - x - 1 = -1$, having solutions x = 0, 1.
For x = 0, $E = 24$ so that x = 0 is a valid solution. For x = 1, $E = 15$ so that x = 1 is **not** a solution.
The first part of condition (iii) gives:
$x^2 - 10x + 24 = 0$, having solutions x = 4, 6.
These are both valid solutions, since neither makes $B = 0$.

Team Project 1998 School:____________________

Name 1:____________________ Name 2:____________________

Name 3:____________________ Name 4:____________________

Team Project – Instructions

1. Print school name and names of contestants neatly in the spaces provided above.

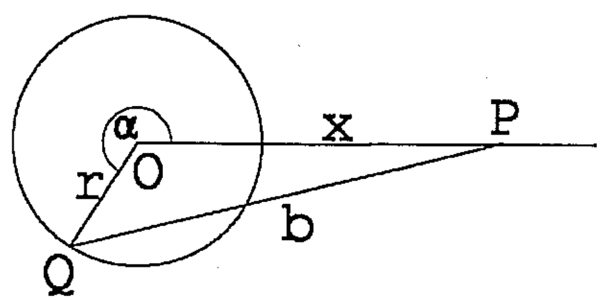
2. You may do your scratch work on the tablecover. Use the color marker provided.

3. Write (or print) the complete solution, **including work required to get your answer**, neatly on the lined paper provided.

4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet as a cover sheet. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be written neatly, using correct English.

5. Good luck and good Mathematics!

Team Project – Problem



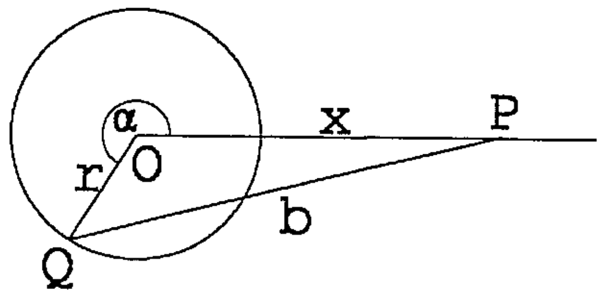
The point Q moves around a circle with center O and radius r. The point P moves along a line through O so that the distance PQ is equal to the constant b. Let α denote the angle POQ and x denote the distance OP.

(a) Find a formula for the distance x in terms of r, b, and α.

(b) In the figure above, b is greater than r. Must this always be true? Explain. What happens if $b = r$?

Solution, Team Project 1998

The point Q moves around a circle with center O and radius r. The point P moves along a line through O so that the distance PQ is equal to the constant b. Let α denote the angle POQ and x denote the distance OP.



(a) We set up an xy-coordinate system with origin at O, positive x-axis in the direction of OP, and y-axis pointing upward. With this setup, the point P will have coordinates (x, 0) and the point Q will have coordinates (r cos α, r sin α). Since the distance PQ is b, we apply the distance formula to obtain

$$(x - r\cos\alpha)^2 + (0 - r\sin\alpha)^2 = b^2$$

then simplifying to obtain

$$x^2 - (2r\cos\alpha)x + r^2 - b^2 = 0$$

solving with the quadratic formula

$$x = \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha - 4(r^2 - b^2)}}{2} = r\cos\alpha \pm \sqrt{r^2(\cos^2\alpha - 1) + b^2}$$

Since x is a distance, it must be positive, we eliminate the negative sign given by the quadratic formula. Thus:

$$x = r\cos\alpha + \sqrt{b^2 - r^2(1 - \cos^2\alpha)} = r\cos\alpha + \sqrt{b^2 - r^2\sin^2\alpha}$$

Note: it is possible to get this result from the law of cosines, provided that you show that cos(α) = cos(π - α).

(b) If we want angle α to vary freely $(0 \le \alpha \le 2\pi)$, we must have $b \ge r$ in order for the expression

$$b^2 - r^2\sin^2\alpha$$

(inside the radical) to always be positive. With some restrictions on angle α, the distance b could be smaller than r. If r = b, we may have x = 2 r cos α until such time as cos α = 0. Then, x = 0 thereafter.

Name:__________________________________ Elementary 2004 School:__________ # 1

A dietician is preparing a meal consisting of foods X, Y, and Z. Each ounce of food X contains 2 units of protein, 3 units of fat, and 4 units of carbohydrate. Each ounce of food Y contains 3 units of protein, 2 units of fat, and 1 unit of carbohydrate. Each ounce of food Z contains 3 units of protein, 3 units of fat, and 2 units of carbohydrate. If the meal must provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?

Name:______________________________ Elementary 2004 School:_________ # 2

The **Preposterous Pizza Place** sells a **Fear Factor Pizza**. Possible toppings are: anchovies, bacon, cauliflower, duck, eggplant, fish, garbanzos, and/or ham. You may have any number of these toppings, or no topping at all. How many possible **Fear Factor Pizzas** are there?

Name:________________________________ Elementary 2004 School:_________ # 3

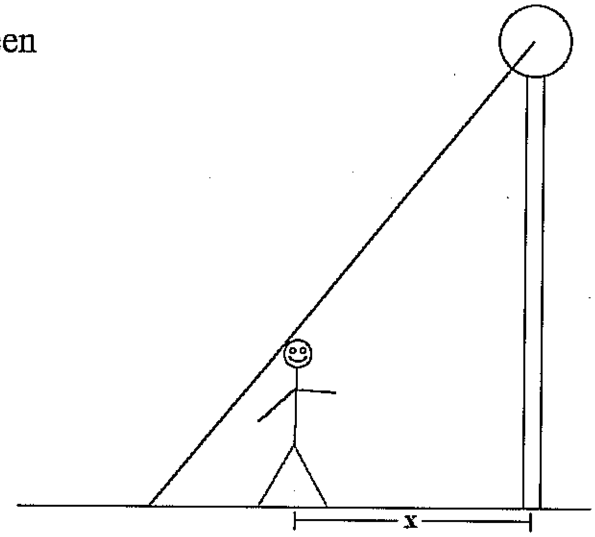
Divide $3x^{10} - 6x^9 + 4x^6 + 3x^5 - 9x^3 + 2x^2 + 16$ by $x - 2$. Find the quotient and remainder.

Name:________________________________ Elementary 2004 School:_________ # 4

Myrtle weeds the garden in an hour and a half. Ivy weeds the garden in 75 minutes. How long will it take to weed the garden if Myrtle and Ivy work together? Give your answer in minutes and seconds – rounded to the nearest second.

Name:________________________________ Elementary 2004 School:__________ # 5

A six-foot tall man is standing x feet away from a streetlight that is eighteen feet tall. Express the distance from the base of the streetlight to the tip of his shadow in terms of x.



Hints and Solutions for Elementary Test Problems 2004

#1 We let A denote the number of ounces of food X to be used in the meal; B the ounces of Y; and C the ounces of Z. We get the system of equations:

$$\begin{cases} 2A+3B+3C=25 \\ 3A+2B+3C=24 \\ 4A+B+2C=21 \end{cases}$$

Using standard methods, we may obtain the solutions:

$$A = 16/5 = 3.2 \text{ oz.}; \quad B = 21/5 = 4.2 \text{ oz.}; \quad C = 2 \text{ oz.}$$

#2 The number of possible pizzas is the number of subsets of a set containing 8 elements. It is $2^8 = 256$ pizzas.

#3 Either Synthetic Division or Long Division will give:

Quotient = $3x^9 + 4x^5 + 11x^4 + 22x^3 + 35x^2 + 72x + 144$ and Remainder = 304.

#4 Let A denote the area of the garden. The rates at which Myrtle and Ivy weed the garden are A/90 and A/75 (units of area per minute) respectively. If T is the time to weed the garden for Myrtle and Ivy working together, we have:

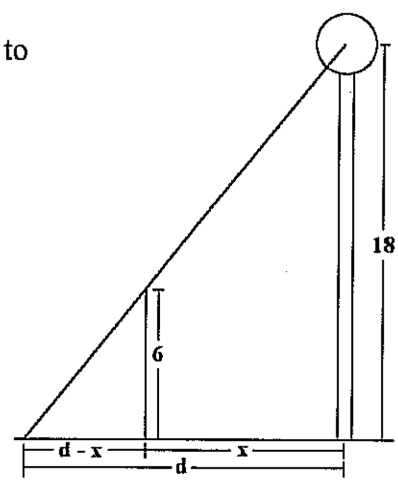
$$T\left(\frac{A}{90}+\frac{A}{75}\right)=A \text{ or } T\left(\frac{1}{90}+\frac{1}{75}\right)=1.$$ Solving for T gives

$$T=\frac{450}{11}=40.909090\cdots \text{ min.}$$

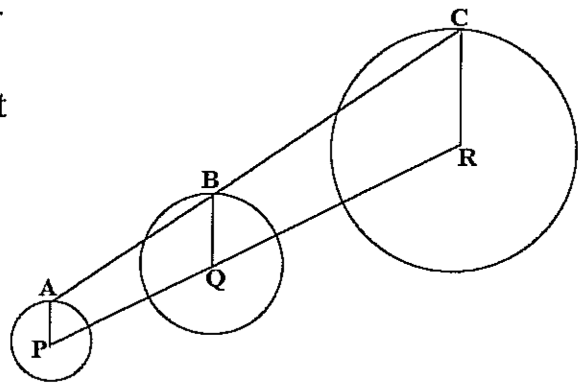
or T ≈ 40 min. 55 sec.

#5 We let d denote the distance from the base of the streetlight to the end of the man's shadow. By similar triangles, we have:

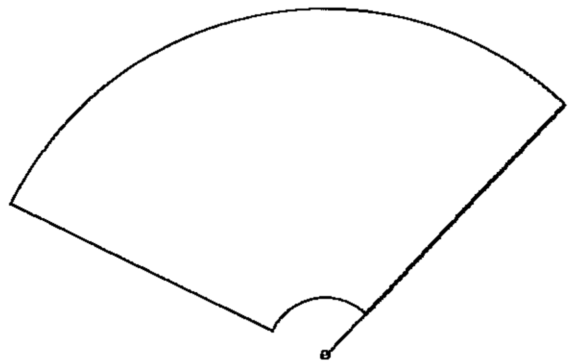
$$\frac{d-x}{6}=\frac{d}{18}.$$ Solving for d gives $d=\frac{3x}{2}$.



The day after a lunar eclipse, Farmer Brown discovered three crop circles in his corn field. (See the figure at the right – not necessarily drawn to scale.) With the help of his farm hand, Farmer Brown began to document this phenomenon. There were strange marks at points A, B, and C, one on each circle. Points A, B, and C were collinear. The centers of the circles, denoted by P, Q, and R, were also collinear. Farmer Brown measured the radii PA and QB to be 15 yards and 40 yards, respectively. He measured the distances PQ and QR to be 100 yards and 150 yards, respectively. He also discovered that the lines PA, QB, and RC were parallel. It was chore time and he didn't get the radius RC measured. Find distance RC.



The wiper blade on Jan's car is 16 inches long and wipes an area bounded by concentric, circular arcs and lines passing through the center of these arcs. The large circular arc has radius 20 inches and the blade sweeps through an angle of $130°$ on a flat window. Find the area covered by this wiper blade to the nearest hundredth of a square inch.



Name:______________________________ Advanced 2004 School:__________ # 3

Find all solutions to the system: $\begin{cases} xy = 3x + 4 \\ x^2 + y^2 = 6y + 1 \end{cases}$

Name:________________________________ Advanced 2004 School:__________ # 4

Solve for θ. Find all exact solutions in radians, with $0 \le \theta < 2\pi$.

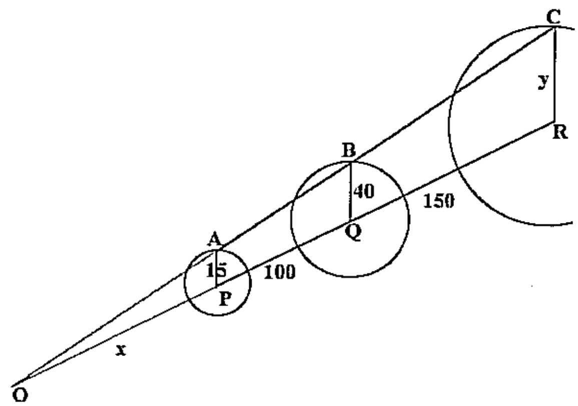
$\sin(4\theta) + \cos(2\theta) = 0$

Name:____________________________________ Advanced 2004 School:__________ # 5

The partial fraction decomposition for the rational expression $\frac{5x^2+4x+2}{x^3+x}$ is $\frac{2}{x}+\frac{3x+4}{x^2+1}$. Find the partial fraction decomposition for the following rational expression: $\frac{7x^2+3x-5}{x^3-x^2+4x-4}$

Hints and solutions for Advanced Test Problems 2004

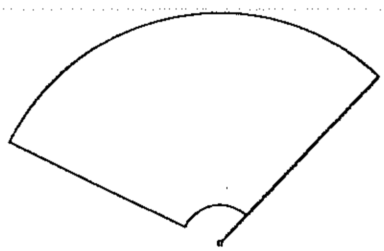
#1 Extend segments AC and PR until they meet at O. Triangles OPA, OQB, and ORC are similar. Letting $x = OP$ and $y = RC$, it follows that:



$$\frac{x}{15} = \frac{x+100}{40} = \frac{x+250}{y}.$$

Solving the first equation for x gives $x = 60$ yds. Substituting this value in the second equation and solving for y gives (the radius RC) $y = 77.5$ yds.

#2 We have two circular sectors – the large one with radius 20 inches and the small one with radius 4 inches. We remember that the area of a circular sector is



$$A = \frac{1}{2}r^2\theta$$

where r is the radius of the sector and θ is the central angle (in radians) of the sector. In this case, the central angle is $\theta = \dfrac{13\pi}{18}$. Subtracting the area of the small sector from the area of the large sector gives the desired area. $\dfrac{1}{2}\left(20^2 - 4^2\right)\dfrac{13\pi}{18} = \dfrac{416\pi}{3} \cong 435.63\,\text{in}^2$

#3 Solving the first equation for y gives: $y = \dfrac{3x+4}{x}$. (*)

Substituting in the second equation, clearing fractions, and simplifying gives the equation:

$x^4 - 10x^2 + 16 = 0$, which has solutions $x = \pm\sqrt{2}$ $\quad x = \pm 2\sqrt{2}$.

Putting each of these four values back in (*), gives the four points:

$\left(\sqrt{2}, 3+2\sqrt{2}\right)$, $\left(-\sqrt{2}, 3-2\sqrt{2}\right)$, $\left(2\sqrt{2}, 3+\sqrt{2}\right)$, $\left(-2\sqrt{2}, 3-\sqrt{2}\right)$

Hints and solutions for Advanced Test Problems 2004

#4 Making the substitution $\varphi = 2\theta$, the equation becomes $\sin(2\varphi) + \cos\varphi = 0$, where we are now looking for solutions satisfying $0 \le \varphi < 4\pi$. Using a double angle formula and factoring gives $\cos\varphi(2\sin\varphi + 1) = 0$.

Setting the first factor equal to zero gives: $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

Setting the second factor equal to zero gives: $\varphi = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$.

Dividing by two gives the values:

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$.

#5 We factor the denominator: $x^3 - x^2 + 4x - 4 = (x-1)(x^2+4)$

The partial fraction decomposition is of the form:

$\frac{7x^2 + 3x - 5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+4}$. Clearing fractions and simplifying gives:

$7x^2 + 3x - 5 = (A+B)x^2 + (C-B)x + (4A - C)$. Equating coefficients gives the three equations: $\begin{cases} A + B = 7 \\ C - B = 3 \\ 4A - C = -5 \end{cases}$, whose solution is:

(A, B, C) = (1, 6, 9). Thus, the partial fraction decomposition is:

$$\frac{1}{x-1} + \frac{6x+9}{x^2+4}.$$

Team Project 2004 School:____________________

Name 1:____________________ Name 2:____________________

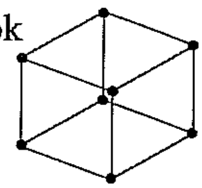
Name 3:____________________ Name 4:____________________

Team Project - Instructions

1. **Print** school name and names of contestants neatly in the spaces provided above.
2. Use unlined paper for scratch work.
3. Write (or print) the complete solution, **including work required to get your answer**, neatly on the lined paper provided. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be written neatly, using correct English.
4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet as a cover sheet
5. Good luck and good Mathematics!

Team Project

Alice is telling Betty about her attempt to get into the Guinness Book of World Records.

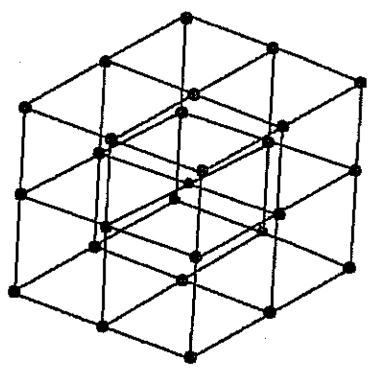


1 x 1 x 1

Alice: "I am making a big, cubic lattice from toothpicks and I need to know how many toothpicks to buy."

Betty: "Will you hold them together with gumdrops?"

Alice: "Maybe – but I thought drops of glue might be cheaper, sturdier, and I wouldn't be tempted to eat it. Here, I made a **one by one by one** (1 x 1 x 1) cubic lattice and a **two by two by two** (2 x 2 x 2) cubic lattice."



2 x 2 x 2

Betty: "I see. The 1 x 1 x 1 lattice took 12 toothpicks and the 2 x 2 x 2 lattice took 54 toothpicks. I suppose you want a formula for the number of toothpicks needed to make an n x n x n lattice."

Alice: "That's the idea."

Your task is to answer the following:

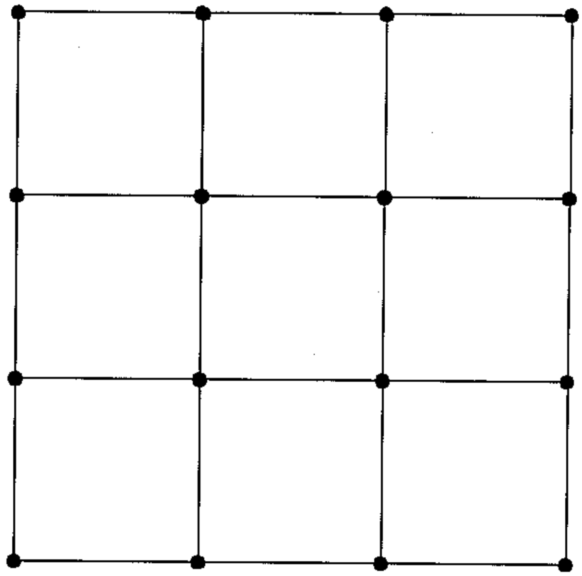
(a) How many toothpicks will be needed for a 3 x 3 x 3 lattice?

(b) Find a formula for the number of toothpicks needed for an n x n x n lattice. Also, give a reasonable justification as to why your formula is correct.

Hints and solution to Team Project 2004

(a) How many toothpicks will be needed for a 3 x 3 x 3 lattice?

Orient the lattice so the edges are aligned north-south, east-west, and perpendicular to those directions. We count the number of east-west toothpicks in one layer to be 12. There are 4 layers. So, there are 48 east-west toothpicks. Multiplying by 3, for the three different directions, gives 144 toothpicks in the lattice.



One layer of a 3 x 3 x 3 lattice

(b) Find a formula for the number of toothpicks needed for an n x n x n lattice. Also, give a reasonable justification as to why your formula is correct.

The formula for the n x n x n case is obtained in a way similar to the 3 x 3 x 3 case. Orient the lattice so the edges are aligned north-south, east-west, and perpendicular to those directions. We count the number of east-west toothpicks in one layer to be

$n(n + 1)$. There are $(n + 1)$ layers. So, there are $n(n+1)^2$ east-west toothpicks.

Multiplying by 3, gives $3n(n+1)^2$ toothpicks in the lattice.

Name:_________________________________ Advanced 2007

School:_________________________________ # 1

We let $\log(r) = \log_{10}(r)$ = the logarithm to the base ten of r. Simplify the following expression.

$$\log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \log\left(\frac{4}{5}\right) + \cdots + \log\left(\frac{198}{199}\right) + \log\left(\frac{199}{200}\right)$$

Name:________________________________ Advanced 2007

School:________________________________ # 2

Find the exact values of k and $\tan\varphi$ so that:

$$3\sin\alpha + 5\cos\alpha = k\sin(\alpha + \varphi)$$

for all angles .

Name:________________________________ Advanced 2007

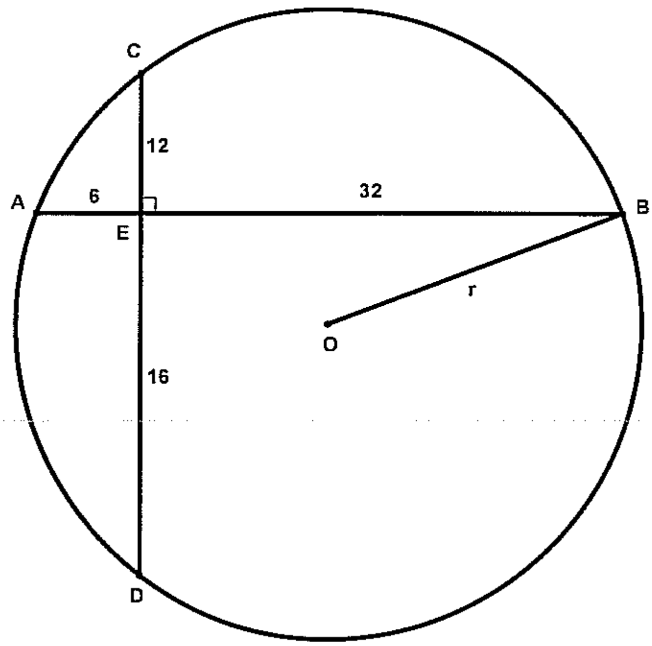
School:________________________________ # 3

Two sides of a triangle have lengths of 3 cm and 7 cm, and meet at an angle of 60°. Find the exact area of the triangle.

Name:__________________________________ Advanced 2007

School:__________________________________ # 4

Two chords, $\overline{AB}$ and $\overline{CD}$ of a circle having center O are perpendicular at point E, with AE = 6, EB = 32, CE = 12, and ED = 16. (The illustration at the right is not necessarily drawn to scale.)



Find the radius r of circle O.

Name:______________________________ Advanced 2007

School:______________________________ # 5

The **Three Men and a Ladder** remodeling company sent Larry, Moe, and Curly to paint your house. Larry works 1.5 times as fast as Moe. Curly works twice as fast as Moe. Together the three men paint your house in eight hours. How long would it take Larry to paint your house, if he had to do it by himself?

Hints and Solutions for the Advanced Contest

#1 Using properties of the logarithm, we can convert the sum into the log of a product:

$$\log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \log\left(\frac{4}{5}\right) + \cdots + \log\left(\frac{198}{199}\right) + \log\left(\frac{199}{200}\right) =$$

$$\log\left(\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{198}{199}\cdot\frac{199}{200}\right) = \log\left(\frac{2}{200}\right) = \log\left(\frac{1}{100}\right) = -2$$

#2 We remember the identity:

(*) $\sin(\alpha+\varphi) = \cos\varphi\sin\alpha + \sin\varphi\cos\alpha$. Since we want

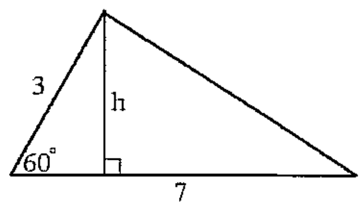
$\sin(\alpha+\varphi) = \frac{3}{k}\sin\alpha + \frac{5}{k}\cos\alpha$, we may reconcile these two equations by letting

$\cos\varphi = \frac{3}{k}$ *and* $\sin\varphi = \frac{5}{k}$. This implies that $\left(\frac{3}{k}\right)^2 + \left(\frac{5}{k}\right)^2 = 1$ and

$k = \sqrt{34}$ *or* $k = -\sqrt{34}$. Finally, we have $\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{5}{3}$.

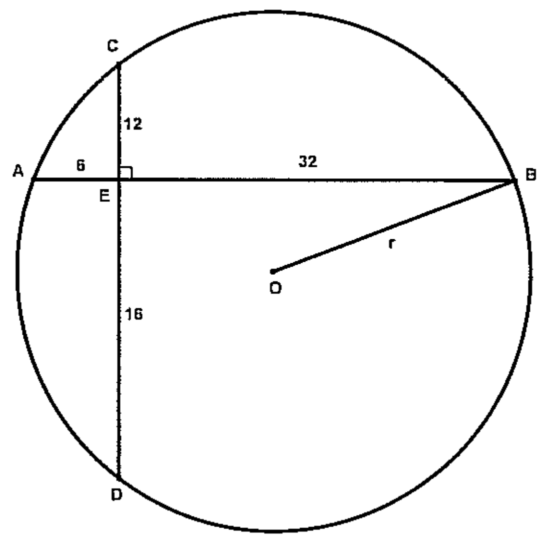
#3 We draw a rough sketch of the triangle. The area is

$$A = \frac{1}{2}(base)\cdot h = \frac{1}{2}\cdot 7\cdot 3\sin(60^o) = \frac{21\sqrt{3}}{4}$$



#4 We may set up a coordinate system with EB as the positive x-axis and EC as the positive y-axis. The center of the circle will lie on the perpendicular bisector of chord AB and also on the perpendicular bisector of chord CD. Thus, the coordinates of the center are:
(13, – 2). It is a simple matter to find the radius.

$$r = \sqrt{365}$$



#5 Let M = the rate at which Moe paints (in house per hour.) We have

$$(1.5M + M + 2M)\cdot 8 = 1, \text{ or}$$

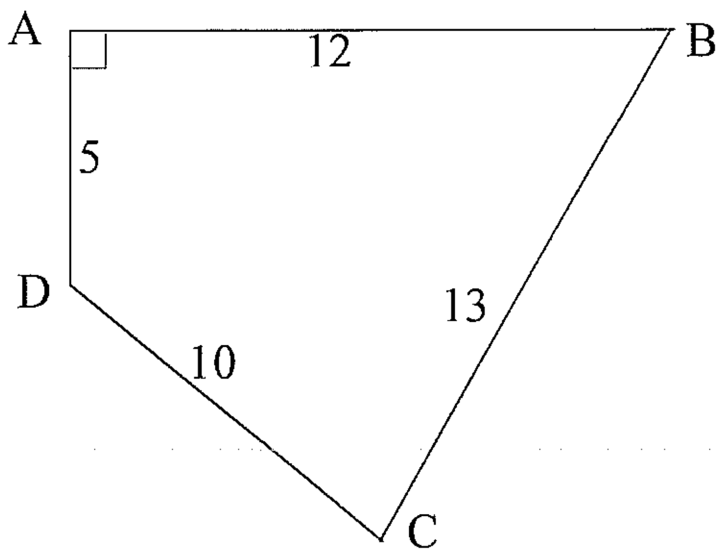
$$M = \frac{1}{36}.$$

It would take Larry 24 hours to paint your house.

Name:______________________________ Elementary 2007

School:______________________________ # 1

The quadrilateral ABCD has sides AB = 12, BC = 13, CD = 10, and DA = 5; with a right angle at vertex A. Find the area of ABCD.



Name:________________________________ Elementary 2007

School:_______________________________ # 2

My coin jar contains only dimes, nickels, and pennies. Thirty-three of the coins are dimes, twenty-five percent of the coins are nickels, and 4/9 of the coins are pennies. How much money is in my coin jar?

Name:________________________________ Elementary 2007

School:________________________________ # 3

What is the greatest prime factor of $55^{100} + 55^{101} + 55^{102}$?

Name:_________________________________ Elementary 2007

School:_________________________________ # 4

In each part below, you are given the first five terms of a sequence $\{a_n\}$ starting with n = 1. Assume that the pattern continues. Write the next two terms and find a formula for the nth term. For example, if you are given $\left\{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \cdots\right\}$, your answers could be:

$a_6 = \frac{-6}{7}$, $a_7 = \frac{7}{8}$, $a_n = \frac{(-1)^{n+1}n}{n+1}$.

(i) $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \frac{5}{14}, \cdots\right\}$

$a_6 =$ $a_7 =$ $a_n =$

(ii) $\left\{\frac{2}{9-4}, \frac{4}{9-16}, \frac{8}{25-16}, \frac{16}{25-36}, \frac{32}{49-36}, \cdots\right\}$

$a_6 =$ $a_7 =$ $a_n =$

Name:______________________________ Elementary 2007

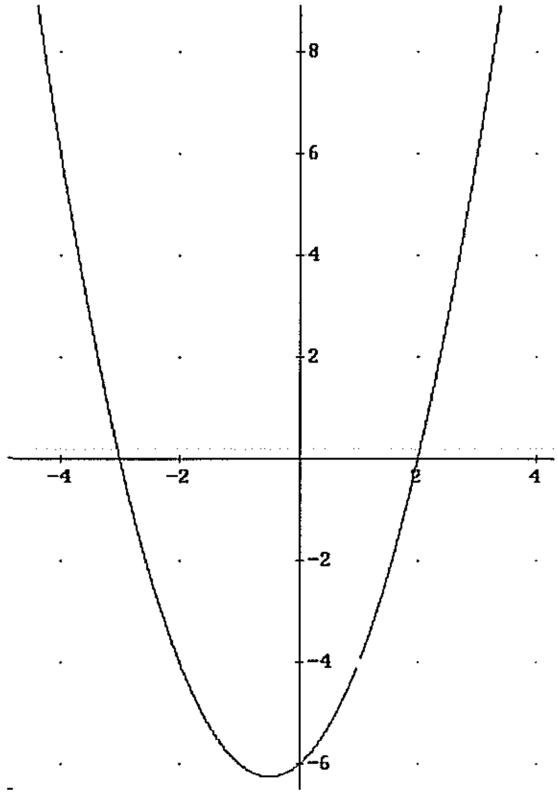
School:______________________________ # 5

Jack is confused. "The graph of a rational function should have a vertical asymptote at each x-value that makes the denominator zero. However, when I plot the graph of

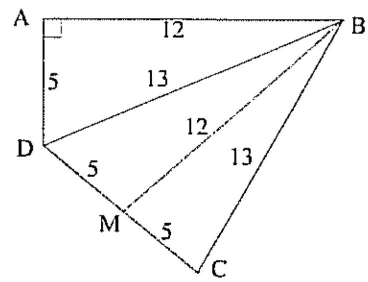
$$y = \frac{x^3 - 7x + 6}{x - 1}$$

with my CAS, there is no vertical asymptote at x = 1. The graph looks like a parabola. How can this be?"

Write a few sentences explaining why this "phenomenon" happened.



Hints and Solutions for the Elementary Contest



#1 We draw the construction line BD. By the Pythagorean Theorem, BD = 13. (Note: the figure may not be drawn to scale.) We let M be the midpoint of side CD and draw construction line BM. We determine the length BM = 12. The quadrilateral is composed of three congruent triangles, each having area 30 square units. The area of ABCD is 90 square units.

#2 Let D = the number of dimes, N = the number of nickels, and P = the number of pennies. We have: D = 33 and the equations

$$\begin{cases}(33+N+P)\cdot 0.25 = N \\ (33+N+P)\cdot \dfrac{4}{9} = P\end{cases}$$

Solving, gives N = 27 and P = 48. So, the amount of money in the jar is: 10 D + 5 N + P = 330 + 135 + 48 = 513 cents or $5.13.

#3 We factor 55^{100} from the expression to get:

$$55^{100}+55^{101}+55^{102} = (1+55+3025)\cdot 55^{100}$$
$$= (3081)\cdot 5^{100}\cdot 11^{100} = 1027\cdot 3\cdot 5^{100}\cdot 11^{100}$$
$$= 13\cdot 79\cdot 3\cdot 5^{100}\cdot 11^{100}$$

Hence, 79 is the greatest prime factor.

#4 For (i) $\left\{\frac{1}{2},\frac{2}{5},\frac{3}{8},\frac{4}{11},\frac{5}{14},\cdots\right\}$, we have: $a_6 = \frac{6}{17}, a_7 = \frac{7}{20},\cdots a_n = \frac{n}{3n-1}$.

For (ii) $\left\{\frac{2}{9-4},\frac{4}{9-16},\frac{8}{25-16},\frac{16}{25-36},\frac{32}{49-36},\cdots\right\}$, we may write the sequence:

$\left\{\frac{2}{3^2-2^2},\frac{-4}{4^2-3^2},\frac{8}{5^2-4^2},\frac{-16}{6^2-5^2},\frac{32}{7^2-6^2},\cdots\right\}$. We have:

$$a_6 = \frac{-64}{8^2-7^2} = \frac{-2^6}{(8-7)(8+7)} = \frac{-2^6}{15},\ a_7 = \frac{2^7}{17},\cdots a_n = \frac{(-1)^{n+1}2^n}{2n+3}$$

#5 The numerator of the rational function: $y = \frac{x^3-7x+6}{x-1}$ is divisible by (x – 1). That is, $y = \frac{x^3-7x+6}{x-1} = \frac{(x-1)(x^2+x-6)}{(x-1)}$, which is equal to $y = x^2+x-6$ for x not equal to one. So, the graph of the rational function is a parabola – with the point (1, – 4) missing.

Team Project 2007 School: ______________________________

Name 1: ______________________ Name 2: ______________________

Name 3: ______________________ Name 4: ______________________

Team Project Instructions

1. **Print** school name and names of contestants neatly in the spaces provided above.
2. Use unlined paper for scratch work.
3. Write (or print) the complete solution, **including work required to get your answer**, neatly on the lined paper provided. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be written neatly, using correct English.
4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet clipped to it as a cover sheet.
5. Good luck and good Mathematics!

Read the conversation between Joe and Jane on this page and the next and then do the Math Team Project.

Joe: "I am remembering the spirit of Archimedes by reproducing one of his great achievements."

Jane: "Will you jump from your bathtub and run through the streets yelling 'EUREKA'?"

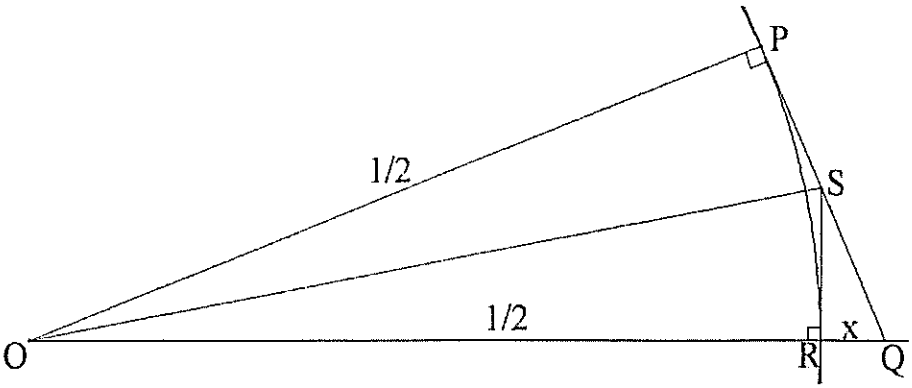
Joe: "No, I want to recreate Archimedes' approximation to . He began with a circle having diameter 1 (one) and created two sets of regular polygons (equal sides and equal interior angles). For one set (the circumscribed polygons), he started with a square, having sides of length 1 (one) that were tangent to the circle at the midpoint of each side. His next polygon was a regular octagon with sides tangent to the circle at the midpoint of each side."

Jane: "So, he got polygons having 4, 8, 16, 32, ... sides. As the number of sides became larger, the perimeter of the polygon approached the circumference of the circle – which is ."

Joe: "I found a **recursion formula** for b_n = the length of one side of the n^{th} circumscribed polygon."

Jane: "OK, you start with $b_1 = 1$, calculate b_2 from b_1, calculate b_3 from b_2, etc. It is a simple matter to find the perimeter of each polygon by multiplying the length of a side by the number of sides. How did you determine the recursion formula?"

Joe: "The figure shows part of a circle with center O having radius ½ and parts of the circumscribed polygons having side lengths b_n and b_{n-1}. In the figure,



we let $x = QR$, $PQ = \dfrac{b_{n-1}}{2}$ and $RS = \dfrac{b_n}{2}$. From the Pythagorean Theorem (applied to OPQ) we have: $\left(x+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{b_{n-1}}{2}\right)^2$ or $2x = \sqrt{(b_{n-1})^2+1} - 1$. Now OPQ is similar to SRQ and $\dfrac{QR}{RS} = \dfrac{QP}{PO}$ giving $b_n = \dfrac{2x}{b_{n-1}} = \dfrac{\sqrt{(b_{n-1})^2+1}-1}{b_{n-1}}$."

Jane: "Neat! The operations required to calculate b_n from b_{n-1} are operations that Archimedes could have performed by hand."

Joe: "Of course, I used my calculator to do the calculations. See, I made a table of the first few approximations to using circumscribed polygons."

n	b_n	number of sides	perimeter
1	1.000000000	4	4.000000000
2	0.414213562	8	3.313708499
3	0.198912367	16	3.182597878
4	0.098491402	32	3.151724906
5	0.049126848	64	3.144118376

Math Team Project

Perform similar work for the regular inscribed polygons having 2^{n+1} sides with vertices on the circle of diameter 1 (one.) In particular:

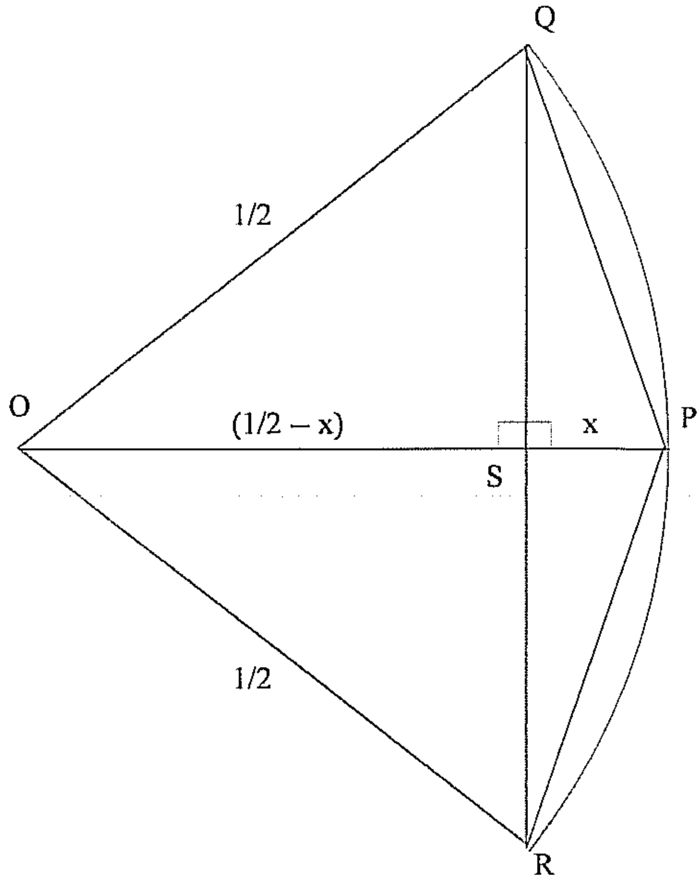
(a) Begin with a square having sides of length $c_1 = \dfrac{\sqrt{2}}{2}$.

(b) Letting c_n denote the length of one side of the n^{th} inscribed polygon, determine a recursion formula for c_n in terms of c_{n-1}.

(c) Calculate the perimeters of the first five inscribed polygons.

Team Project 2007 Hints and Solutions

The figure shows part of a circle with center O having radius ½ and parts of the inscribed polygons having side lengths $PQ = c_n$ and $QR = c_{n-1}$. In the figure, we let $x = SP$. Applying the Pythagorean theorem to triangle OSQ, we get



$$\left(\frac{1}{2}-x\right)^2+\left(\frac{c_{n-1}}{2}\right)^2=\left(\frac{1}{2}\right)^2$$

or

$$x=\frac{1-\sqrt{1-\left(c_{n-1}\right)^2}}{2}.$$

Applying the Pythagorean theorem to triangle PSQ, we get

$$c_n=\sqrt{x^2+\left(\frac{c_{n-1}}{2}\right)^2}$$

replacing x gives the recursion relation:

$$c_n=\sqrt{\frac{1-\sqrt{1-\left(c_{n-1}\right)^2}}{2}}.$$

Below is a table of the first five approximations to using inscribed polygons.

n	c_n	Number of sides	Perimeter
1	0.7071067811	4	2.828427124
2	0.3826834323	8	3.061467458
3	0.1950903220	16	3.121445152
4	0.09801714032	32	3.136548490
5	0.04906767432	64	3.140331157