#1 **Elementary 2013**

Line ℓ has equation $3x - 2y = 5$. Line m intersects ℓ when $x = -3$. Suppose $(1, -4)$ lies on m.

Find the x-coordinate of the point $P(x, 3)$ that also lies on m.

#2 **Elementary 2013**

A rectangular field has a perimeter of 130 meters. The length of the field is 12 meters more than its width. Find the area of the field.

#3 **Elementary 2013**

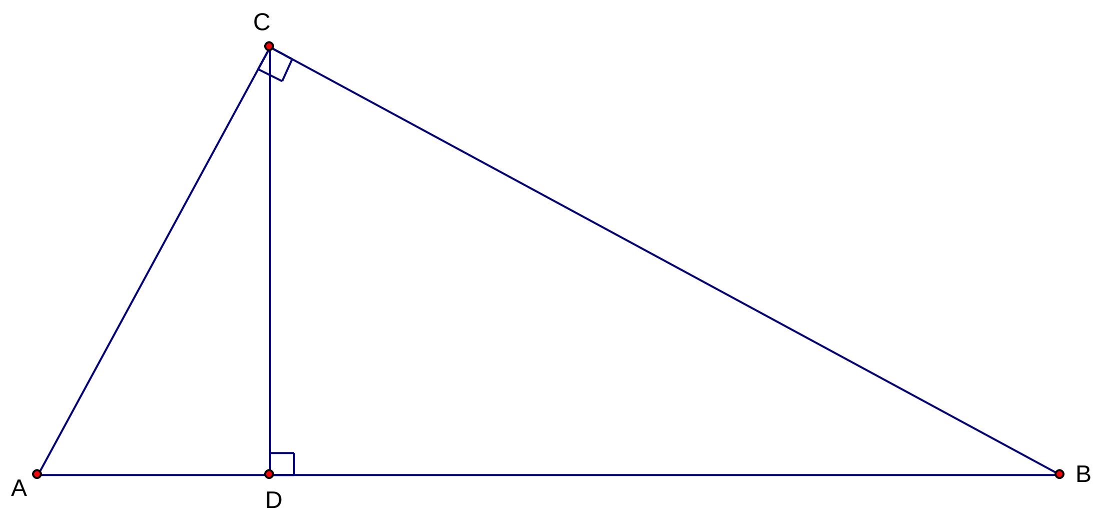
Consider the points $(3, -1)$, $(-2, 14)$, and $(120, 38)$. Are these points the vertices of a right triangle? If so, prove it. If not, show why not.

#4 **Elementary 2013**

Solve the system: $\begin{cases} \sqrt{x+4} = \sqrt{y+3} - 1 \\ 3x - y = 2 \end{cases}$

#5 **Elementary 2013**

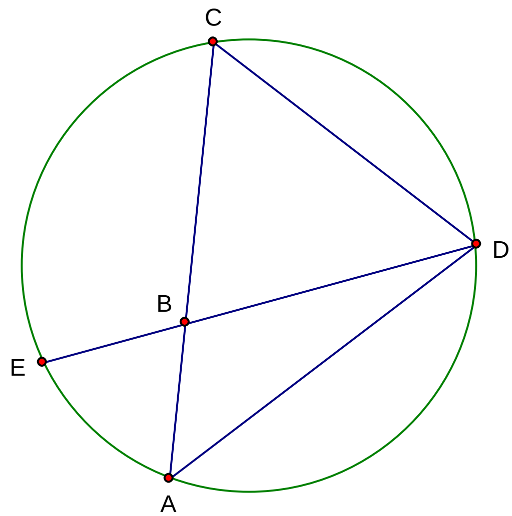
In ΔABC, $AD = 5$ and $AB = 15$. $\overline{CD} \perp \overline{AB}$ and $\overline{AC} \perp \overline{CB}$. Find the exact value of CD.



#1 **Advanced 2013**

In the figure, $AB = 4$, $AD = 11, BC = CD$, $m\widehat{AE} = 40°$, and $m\angle ACD = 40°$.

Find $m\widehat{EC}, m\angle CAD$, and CB. Note: The figure is not drawn to scale.



$m\widehat{EC} =$ ______________

$m\angle CAD =$ ___________

$CB =$ _____________

#2 **Advanced 2013**

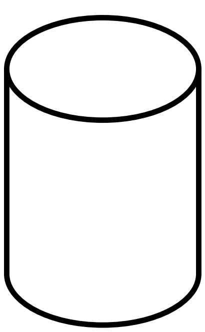
Find the exact value of $\tan\frac{11\pi}{12}$ and write answer in simplest form.

#3 **Advanced 2013**

Solve for x: $\ln(x^{\ln x}) = 4$.

#4 **Advanced 2013**

Find the exact height of a right circular cylinder whose radius is 4 in. and whose total surface area (i.e., bases and lateral surface) is 36 in^2.



#5 **Advanced 2013**

The partial fraction decomposition for the rational expression $\frac{5x^2+4x+2}{x^3+x}$ is $\frac{2}{x}+\frac{3x+4}{x^2+1}$. Find the partial fraction decomposition of $\frac{100x}{(x^2+1)(2x^2-3x-2)}$.

Hints and Solutions - Elementary 2013

1. The intersection of lines $\boldsymbol{\ell}$ and $\boldsymbol{m}$ is (– 3, – 7). So the slope of line $\boldsymbol{m}$ is $\frac{-7-(-4)}{-3-1} = \frac{-3}{-4} = \frac{3}{4}$ and

   an equation for line $\boldsymbol{m}$ is $y + 7 = \frac{3}{4}(x + 3)$. On line $\boldsymbol{m}$, if $y = 3$, then $x = \frac{31}{3}$.

2. Let w = the width of the field. Then $w + 12$ = the length of the field. So $2w + 2(w + 12) = 130$

   and $w = \frac{53}{2}$. Thus the area is $w(w + 12) = \frac{4081}{4} = 1020\frac{1}{4}$ square meters.

3. The distance between vertices (3, – 1) and (– 2, 14) is $a = \sqrt{\left(3-(-2)\right)^2 + (-1-14)^2} =$
$\sqrt{250}$.

   Similarly, the lengths of the other two sides are $b = \sqrt{15210}$ and $c = \sqrt{15460}$. Since $a^2 + b^2 =$
c^2,

   the Pythagorean Theorem is satisfied, and this is a right triangle.

4. Solve the second equation for y. Then substitute $3x - 2$ for y in the first equation: $\sqrt{x+4} =$
$\sqrt{y+3} - 1$.

   Then simplify to get: $\sqrt{x+4} = \sqrt{3x+1} - 1$

   Square both sides, collect like terms, and simplify to obtain: $\sqrt{3x+1} = x - 1$

   Square both sides again, collect like terms, and simplify to obtain: $0 = x^2 - 5x$

   Factor: $0 = x(x-5)$, so either $x = 0$ or $x = 5$.

   If $x = 0$, then $y = -2$, but this does not satisfy the original system.

   If $x = 5$, then $y = 13$. This is the only solution.

5. Since ΔADC and ΔCDB are similar triangles, the ratios of their corresponding sides are equal.

   Thus $\frac{AD}{CD} = \frac{CD}{BD}$. That is, $CD^2 = AD \cdot BD = 5(10) = 50$. So, $CD = 5\sqrt{2}$.

Hints and Solutions - Advanced 2013

1. Since ΔBCD is an isosceles triangle, $m\angle CBD = m\angle CDB$. Since $m\angle ACD = 40°$, $m\angle CBD =$

$m\angle CDB = 70°$. Thus $m\widehat{EC} = 140°$. Since $m\widehat{AE} = 40°$ and $m\widehat{EC} = 140°$, AC is actually a diameter,

so $\angle CDA$ is a right angle. Using the Pythagorean Theorem on ΔADC, we have: $AD^2 + DC^2 = AC^2$.

That is, $11^2 + CB^2 = (CB + 4)^2$ since $CB = DC$. Expanding and simplifying leads to $CB = \frac{105}{8}$.

Since $m\angle ACD + m\angle CDA = 130°$, $m\angle CAD = 50°$.

2. Since $\frac{11\pi}{12} = \frac{\pi}{6} + \frac{3\pi}{4}$, $\tan\frac{11\pi}{12} = \tan\left(\frac{\pi}{6} + \frac{3\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{6}+\frac{3\pi}{4}\right)}{\cos\left(\frac{\pi}{6}+\frac{3\pi}{4}\right)} = \frac{\sin\frac{\pi}{6}\cos\frac{3\pi}{4} + \cos\frac{\pi}{6}\sin\frac{3\pi}{4}}{\cos\frac{\pi}{6}\cos\frac{3\pi}{4} - \sin\frac{\pi}{6}\sin\frac{3\pi}{4}} =$
$\frac{\frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)} = \frac{-\sqrt{2}+\sqrt{6}}{-\sqrt{6}-\sqrt{2}}$.

Rationalizing the denominator gives $\tan\frac{11\pi}{12} = \sqrt{3} - 2$.

3. Using a property of logarithms, $\ln\left(x^{\ln x}\right) = (\ln x)(\ln x)$. So, $(\ln x)^2 = 4$ and $\ln x = \pm 2$. Therefore,

$x = e^2$ or $x = \frac{1}{e^2}$.

4. There are two bases of the cylinder and each has area = $\pi(4)^2 = 16\pi$. The lateral surface may be regarded

as a rectangle with length equal to the circumference of the base of the cylinder = $2\pi(4) = 8\pi$ and width

equal to the height of the cylinder, say h. So the lateral surface area is $8\pi h$.

Then the surface area is $2(16\pi) + 8\pi h = 36$. Thus $h = \frac{36-32\pi}{8\pi} = \frac{9-8\pi}{2\pi}$ in^2.

5. Since $2x^2 - 3x - 2 = (2x+1)(x-2)$, $\frac{100x}{(x^2+1)(2x^2-3x-2)} = \frac{100x}{(x^2+1)(2x+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{2x+1} + \frac{D}{x-2}$.

Thus $(Ax+B)(2x+1)(x-2) + C(x^2+1)(x-2) + D(x^2+1)(2x+1) = 100x$. Substituting 2 for x

yields $D = 8$ and substituting $-\frac{1}{2}$ for x yields $C = 16$. If we substitute 0 for x, 16 for C, and 8 for D, we have $B = -12$. Finally, substituting 1 for x and all of the known values for B, C, and D, we have $A = -16$.

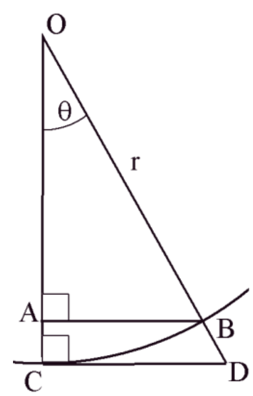
Thus $\frac{100x}{(x^2+1)(2x^2-3x-2)} = \frac{-16x-12}{x^2+1} + \frac{16}{2x+1} + \frac{8}{x-2}$.

Team Project 2013

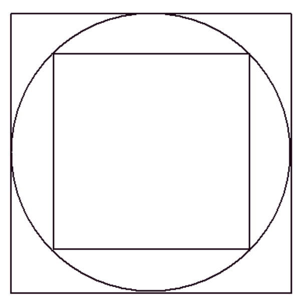
Team Project – Instructions

1. Print school name and names of team members neatly in the spaces provided above.
2. Use unlined paper for scratch work. Use paperclip when returning exam and all pages.
3. Print the complete solution, including work required to get your answer, neatly on the lined paper provided. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be neatly printed and clearly expressed.
4. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet clipped to it as a cover sheet.
5. Good luck and good Mathematics!

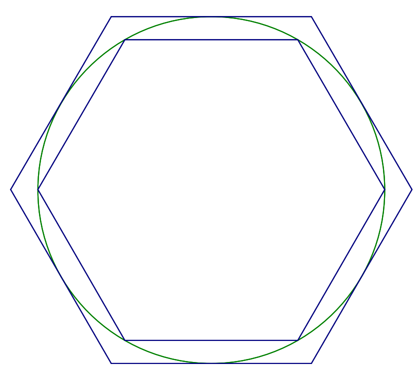
(a) A circle with center O and radius r has right triangles OAB and OCD arranged as in the figure. The circle is tangent to segment CD at C. The radius of this circle is: r = length of OB = length of OC. The angle θ = angle AOB. Express the ratio, ρ = area(OCD)/area(OAB) in terms of trigonometric functions of angle θ. You may wish to use the results of this problem to work parts (b) and (c) below.



(b) Squares are inscribed in, and circumscribed outside a circle – as shown in the figure. If the smaller square has an area of three square inches, what is the area of the larger square?

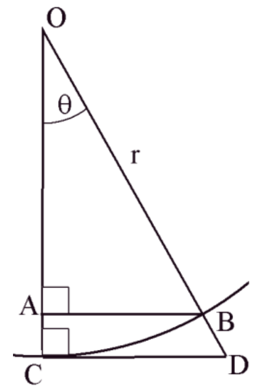


(c) Regular hexagons are inscribed in, and circumscribed outside a circle – as shown in the figure. If the smaller hexagon has an area of three square inches, what is the area of the larger hexagon?



Team Project 2013 – Hints and Solutions

(a) A circle with center O and radius r has right triangles OAB and OCD arranged as in the figure. The circle is tangent to segment CD at C. The radius of this circle is: r = length of OB = length of OC. The angle θ = angle AOB. Express the ratio, ρ = area(OCD)/area(OAB) in terms of trigonometric functions of angle θ. You may wish to use the results of this problem to work parts (b) and (c) below.



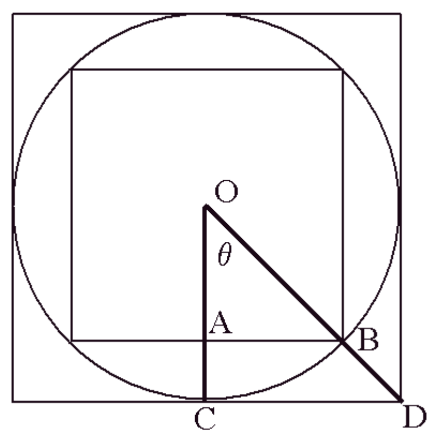
Solution: We have AB = $r \sin\theta$ and OA = $r \cos\theta$, so that the area of

triangle OAB = $\frac{1}{2} r^2 \sin\theta \cos\theta$.

Similarly, OC = r and CD = $r \tan\theta$. The area of triangle OCD = $\frac{1}{2} r^2 \tan\theta$. It follows that the ratio $\rho = \sec^2\theta$.

(b) In the case of the squares, we have $\theta = 45°$ and

$\rho = 2$. The ratio of the area of the larger triangle to the smaller triangle is two. It follows that the ratio of the larger square to the smaller square is also two. Hence the larger square has area = $3 \cdot 2 = 6$ square inches.



(c) In the case of the hexagons, $\theta = 30°$, so that $\sec^2\theta = 4/3$ and the larger hexagon has area = 4 square inches.

#1 **Elementary 2015**

What is the sum of all the roots of the equation:

$$(2x+3)(x-4)+(2x+3)(x-6)=0 \text{ ?}$$

#2 **Elementary 2015**

Sam was asked by his teacher to subtract 3 from a certain number and then divide the result by 9. Instead, he subtracted 9 from the number and then divided the result by 3, giving an answer of 43. What would the answer have been had he worked the problem correctly?

#3 **Elementary 2015**

Sarah pours four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then transfers half of the coffee from the first cup to the second and, after stirring thoroughly, transfers half the mixture in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

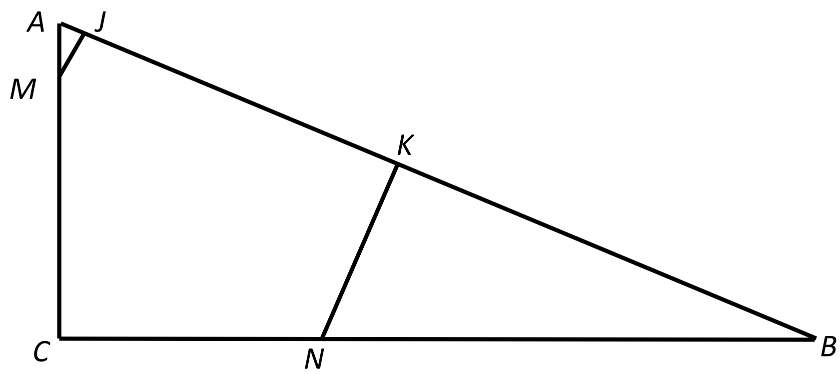
#4 **Elementary 2015**

Describe the graph of the equation: $x^2+y^2=(x+y)^2$.

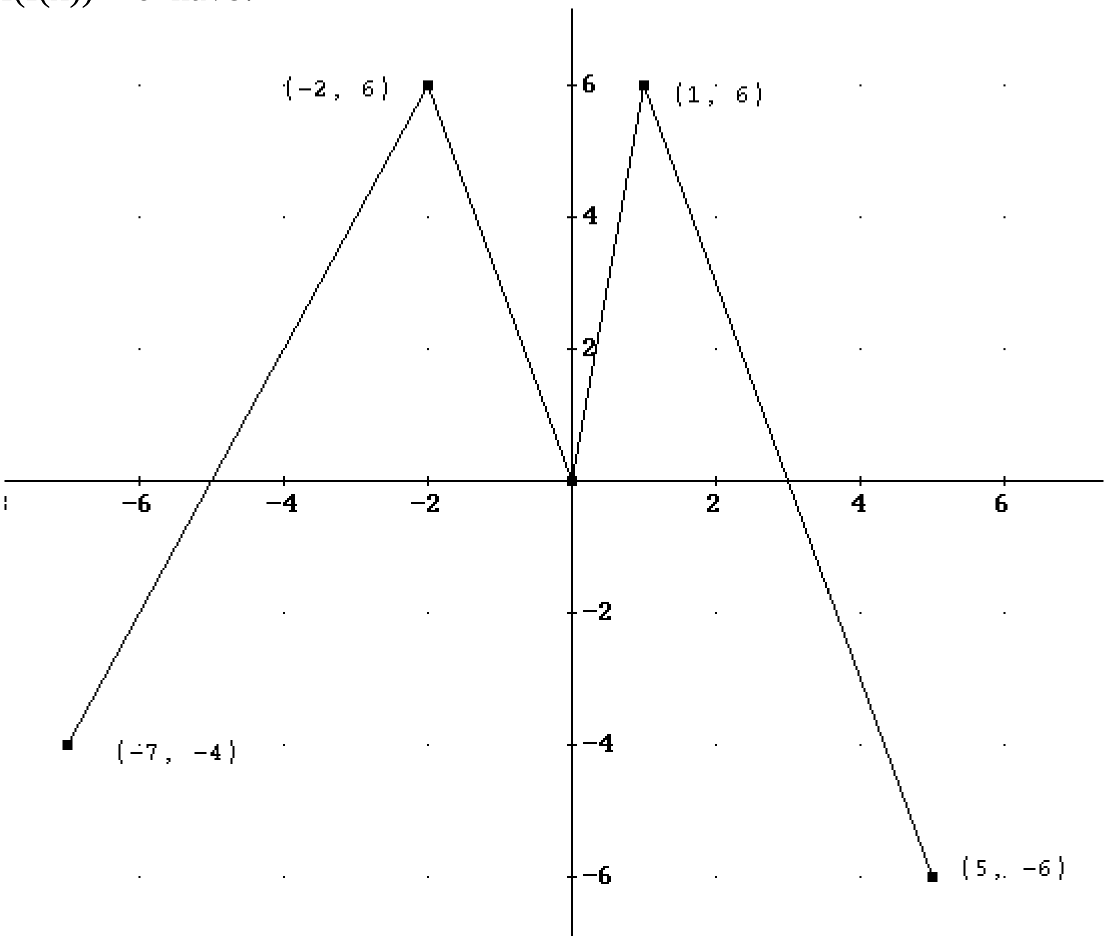
#5 **Elementary 2015**

A positive number x has the property that x% of x is 4. What is x?

In triangle ABC, $AB = 13$, $AC = 5$, and $BC = 12$. Points M and N lie on $\overline{AC}$ and $\overline{BC}$, respectively, with $CM = CN = 4$. Points J and K lie on $\overline{AB}$ so that $\overline{MJ}$ and $\overline{NK}$ are perpendicular to $\overline{AB}$. Find the exact area of pentagon $CMJKN$.



The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 6$ have?



#3 **Advanced 2015**

Suppose that cos(x) = 0 and cos(x + z) = ½. What is the smallest possible positive value for z? Give your answer in radians.

#4 **Advanced 2015**

Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

#5 **Advanced 2015**

(A) For how many values of a is it true that the line $y = x + a$ passes through the vertex of the parabola $y = x^2 + a$?

(B) For how many values of a is it true that the line $y = x + a$ passes through the vertex of the parabola $y = x^2 + a^2$?

Hints 2015 Elementary Contest

1. We may factor the equation:
   $(2x+3)(2x-10)=0$. The roots are $x = -3/2$ and $x = 5$. The sum of these roots is 7/2.

2. We let N denote the starting number. We have: (N – 9)/3 = 43. Solving, gives N = 138, so that (N – 3)/9 = 15.

3. Let us start with cup 1 containing 4 oz coffee and cup 2 containing 4 oz cream. She then transfers half coffee from the first cup to the second so that now the first cup has 2 oz coffee and the second cup has 2 oz coffee mixed with 4 oz cream. Her last transfer takes 1 oz coffee and 2 oz cream from the second cup and adds it to the first. The first cup now has 3 oz coffee and 2 oz cream. So, $\frac{2}{5}$ of the liquid in the first cup is cream.

4. Subtracting x^2+y^2 from the given equation gives $2xy = 0$. Thus $x = 0$ or $y = 0$. The graph is all points on the *x*-axis and the *y*-axis.

5. Since x% of x is 4, $\frac{x}{100}\cdot x = \frac{x^2}{100} = 4$. Solving, gives $x = 20$.

2015 Advanced hints and solutions

1. Triangles *AJM*, *NKB*, and *ACB* are right triangles. ΔAJM is similar to ΔACB, so $\frac{AJ}{AC} = \frac{AM}{AB}$ and $\frac{JM}{BC} = \frac{AM}{AB}$. That is, $\frac{AJ}{5} = \frac{1}{13}$ and $\frac{JM}{12} = \frac{1}{13}$, giving $AJ = \frac{5}{13}$ and $JM = \frac{12}{13}$. The area of ΔAJM is $\frac{1}{2}\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) = \frac{30}{169}$.
   Since ΔNKB and ΔACB are also similar, we can solve the proportions $\frac{NK}{AC} = \frac{NB}{AB}$ and $\frac{BK}{BC} = \frac{NB}{AB}$, giving $NK = \frac{40}{13}$ and $BK = \frac{96}{13}$. The area of ΔNKB is $\frac{1}{2}\left(\frac{40}{13}\right)\left(\frac{96}{13}\right) = \frac{1920}{169}$.
   The area of ΔACB is $\frac{1}{2}(5)(12) = 30$.
   Area of pentagon *CMJKN* = area ΔACB – area ΔAJM – area ΔNKB = $\frac{240}{13}$.

2. From the graph of the function f, we can see that f(w) = 6 can only happen if w = – 2 or w = 1. That is, – 2 = f(x) or 1 = f(x). There will be two values, x, where – 2 = f(x) and four values, x, where 1 = f(x). Hence, f(f(x)) = 6 has: 2 + 4 = 6 solutions.

3. From the identity: cos(x + z) = cos(x) cos(z) – sin(x) sin(z), we determine ½ = 0 cos(z) – sin(x) sin(z). Since cos(x) = 0, we have sin(x) = ± 1, so that $\pm$½ = sin(z) and z = $\pi/6$.

4. From the definition, $h \otimes h = h^3 - h$. It follows that
   $h \otimes (h \otimes h)$ = $h \otimes (h^3 - h) = (h^3) - (h^3 - h) = h$

5. (A) The vertex of the parabola $y = x^2 + a$ is the point (0, a). Every line $y = x + a$ passes through this vertex (infinitely many values of a.)

   (B) The vertex of the parabola $y = x^2 + a^2$ is the point $(0, a^2)$. If the line $y = x + a$ passes through this vertex, we conclude that $a = a^2$. This happens when a = 1 or a = 0 (two values of a.)

Team Project 2015

Team Project – Instructions

6. Print school name and names of team members neatly in the spaces provided above.
7. Use unlined paper for scratch work. Use paperclip when returning exam and all pages.
8. Print the complete solution, including work required to get your answer, neatly on the lined paper provided. Entries will be judged on the mathematical correctness and the organization of the solution. Answers should be neatly printed and clearly expressed.
9. Stay at your table until a proctor collects your solutions. Hand in solutions (on lined paper) with this problem sheet clipped to it as a cover sheet.
10. Good luck and good Mathematics!

Deb and Steve are working on the following problem.

Imelda Spider has 25 shoes in her closet. She must decide on an outfit (consisting of eight shoes) for the Spring Cotillion. How many possible outfits can she assemble?

Steve: Imelda owns 25 shoes – not 25 pairs of shoes?
Deb: Right. Let's call them shoe A, shoe B, shoe C, . . . , shoe Y.
Steve: Do socks count?
Deb: No. There was no mention of any other clothing.
Steve: So, she must select an octuplet of shoes for each possible "outfit."
Deb: Yes, and the order of selection is important.
Steve: Should we give our answer in terms of factorials?
Deb: Yes. That should make our answer more concise.

(a) How many possible outfits could Imelda assemble for the Spring Cotillion?
(b) Suppose she owns n shoes, (n > 8) how many possible outfits could she then assemble?

Team Project 2015 Hints and Solutions

(a) How many possible outfits could Imelda assemble for the Spring Cotillion?
She must select one shoe for each of her eight feet. For the first foot, there are 25 possible shoes.

For foot #2, there will be 24;
for foot #3, 23;
etc.

The number of possible outfits is:
$(25)(24)(23) \cdot \cdot (18) = \frac{25!}{17!}$

(b) An analogous argument for the case that she has n shoes gives:
$(n)(n-1)(n-2) \cdot \cdot (n-7) = \frac{n!}{(n-8)!}$ possible outfits.